On the critical case of Okamoto's continuous non-differentiable functions

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Abstract: In a recent paper in this Proceedings, H. Okamoto presented a parameterized family of continuous functions which contains Bourbaki's and Perkins's nowhere differentiable functions as well as the Cantor-Lebesgue singular function. He showed that the function changes it's differentiability from 'differentiable almost everywhere' to 'non-differentiable almost everywhere' at a certain parameter value. However, differentiability of the function at the critical parameter value remained unknown. For this problem, we prove that the function is non-differentiable almost everywhere at the critical case.

Key words: Continuous non-differentiable function; the law of the iterated logarithm.

1. Introduction. We consider a parameterized family of continuous functions which were presented by H. Okamoto [3, 4]. This function can be regarded as a generalization of Bourbaki's [1] and Perkins's [5] nowhere differentiable functions as well as of the Cantor-Lebesgue singular function.

Okamoto's function is constructed as the limit of a sequence $\{f_n\}_{n=0}^{\infty}$ of piecewise linear and continuous functions. For a fixed parameter $a \in (0, 1)$, each function in the sequence is defined as follows:

(i) $f_0(x) = x$,

(ii) $f_{n+1}(x)$ is continuous on [0,1],

(iii)
$$f_{n+1}\left(\frac{k}{3^n}\right) = f_n\left(\frac{k}{3^n}\right),$$
$$f_{n+1}\left(\frac{3k+1}{3^{n+1}}\right) = f_n\left(\frac{k}{3^n}\right) + a\left[f_n\left(\frac{k+1}{3^n}\right) - f_n\left(\frac{k}{3^n}\right)\right],$$
$$f_{n+1}\left(\frac{3k+2}{3^{n+1}}\right) = f_n\left(\frac{k}{3^n}\right) + (1-a)\left[f_n\left(\frac{k+1}{3^n}\right) - f_n\left(\frac{k}{3^n}\right)\right],$$
$$f_{n+1}\left(\frac{k+1}{3^n}\right) = f_n\left(\frac{k+1}{3^n}\right),$$
$$\text{for } k = 0, 1, \cdots, 3^n - 1,$$

(iv) $f_{n+1}(x)$ is linear in each subinterval
$$\frac{k}{3^{n+1}} \le x \le \frac{k+1}{3^{n+1}} \quad \text{for} \quad k = 0, 1, \cdots, 3^{n+1} - 1.$$

Figure 1 shows the operation from f_n to f_{n+1}. Okamoto's function $F_a(x)$ is then defined as
$$F_a(x) = \lim_{n\to\infty} f_n(x).$$
He noticed that $F_a(x)$ is continuous on $[0, 1]$ and coincides with some known functions when a takes particular values. For example, the cases $a = 5/6$ and $a = 2/3$ correspond to nowhere-differentiable functions defined by Perkins [5] and Bourbaki [1] respectively. Also, if $a = 1/2$, F_a is the Cantor-Lebesgue singular function which is non-decreasing and has zero derivative almost everywhere (Fig. 2).

2. Differentiability of F_a. In the paper [3], H. Okamoto proved that $F_a(x)$ has the following features:

(i) If $a < a_0$, then $F_a(x)$ is differentiable almost everywhere.

(ii) If $a_0 < a < 2/3$, then $F_a(x)$ is non-differentiable almost everywhere.

(iii) If $2/3 \le a < 1$, then $F_a(x)$ is nowhere differentiable.

Here, the constant $a_0(= 0.5592\cdots)$ is the unique real root of
$$54a^3 - 27a^2 = 1.$$

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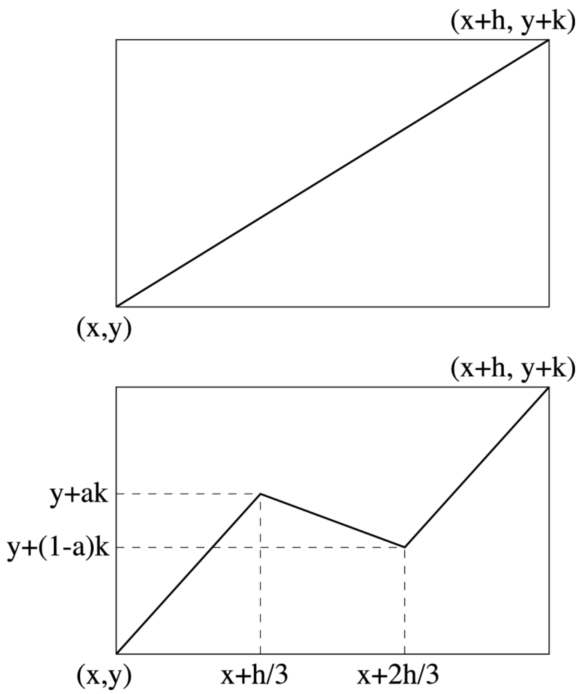

Fig. 1. The operation from f_n to f_{n+1}. Before the operation (top) and after the operation (bottom). This operation is performed in each subinterval $[k/3^n, (k+1)/3^n]$.

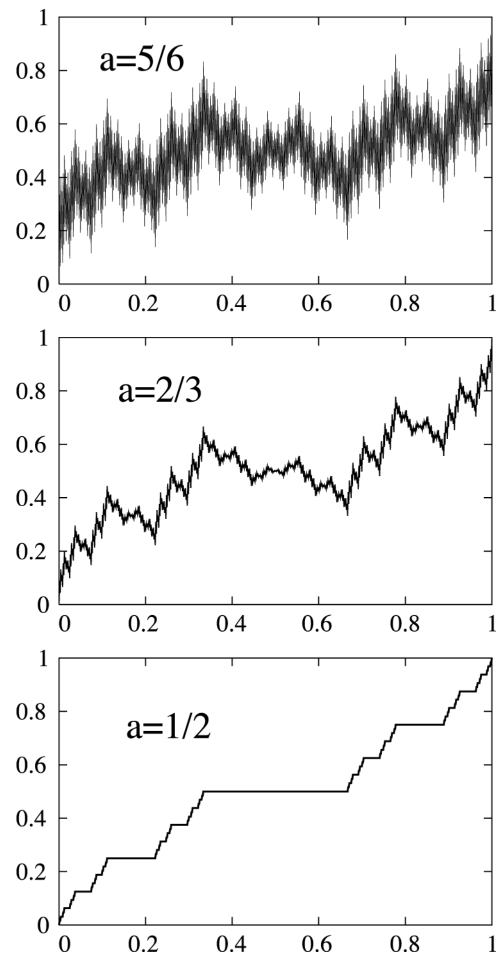

Fig. 2. The graph of Perkins's function (top), Bourbaki's function (middle) and the Cantor-Lebesgue singular function (bottom).

As for the case $a = a_0$, it remained open whether $F_a(x)$ is differentiable almost everywhere or non-differentiable almost everywhere. In this case, we proved that $F_a(x)$ is non-differentiable almost everywhere.

3. Main result. The main result of this article is the following

Theorem 1. *If $a = a_0$, then $F_a(x)$ is non-differentiable almost everywhere in* $[0, 1)$.

In order to prove this theorem, we need some definitions and a preliminary lemma concerning with the law of the iterated logarithm [2].

Definitions. Let

$$x = \sum_{n=1}^{\infty} \frac{\xi_n(x)}{3^n}, \qquad \xi_n(x) \in \{0, 1, 2\},$$

denote the ternary expansion of $x \in [0, 1)$. If x is a rational number of the form $k/3^n$, we use the ternary expansion ending in all 0's (instead of the one ending in all 2's). We also use the following notations:

$$c(k) = \begin{cases} 1, & (k = 0 \quad \text{or} \quad k = 2), \\ -2, & (k = 1), \end{cases}$$

and

$$S_n(x) = \sum_{k=1}^{n} c\Big(\xi_k(x)\Big),$$

$$T_n(x) = 1\Big(\xi_{n+1}(x) = 1\Big) \cdot S_n(x),$$

where $1(A)$ is the indicator function that takes the value one if argument A is true and zero otherwise.

With these definitions, we have the following lemma:

Lemma 1.

$$\limsup_{n \to \infty} \frac{T_n(x)}{\sqrt{n}} \geq 1$$

holds for almost every $x \in [0, 1)$.

Proof. Since the $c(\xi_n)$ are i.i.d. random variables with mean 0 and variance 2 with respect to Lebesgue measure on $(0, 1)$, the law of the iterated logarithm [2] implies that

$$\limsup_{n\to\infty} \frac{S_n(x)}{\sqrt{4n\log\log n}} = 1$$

and

$$\liminf_{n\to\infty} \frac{S_n(x)}{\sqrt{4n\log\log n}} = -1$$

almost everywhere in $(0,1)$.

Thus in particular, the events $S_n(x)/\sqrt{n} \geq 1$ and $S_n(x)/\sqrt{n} \leq -1$ both happen infinitely often. Each time $S_n(x)/\sqrt{n}$ exits the interval $[1,\infty)$, it must do so with a value of

$$c\Big(\xi_{n+1}(x)\Big) = -2$$

(the only negative value). Thus,

$$\frac{T_n(x)}{\sqrt{n}} \geq 1$$

happens infinitely often as well. □

We now complete the proof of the main theorem.

Proof of theorem 1. We first note that $F_a(x)$ has the following representation:

$$F_a(x) = \sum_{k=1}^{\infty} \Psi_k(x),$$

$$\Psi_k(x) = \prod_{l=1}^{k-1} p\Big(\xi_l(x)\Big)\cdot q\Big(\xi_k(x)\Big),$$

where

$$p(0)=a,\quad p(1)=1-2a,\quad p(2)=a,$$

$$q(0)=0,\quad q(1)=a,\quad q(2)=1-a.$$

In what follows, we assume that $a=a_0$ and x satisfies

$$\limsup_{n\to\infty}\frac{T_n(x)}{\sqrt{n}} \geq 1.$$

From the definition of $T_n(x)$, we can take an increasing sequence $\{r_n\}$ which satisfies

$$\sum_{k=1}^{r_n} c\Big(\xi_k(x)\Big) \geq \sqrt{r_n},\quad \xi_{r_n+1}(x)=1,\quad n=1,2,3,\cdots.$$

Here we define $\{x_n\}$ by

$$x_n = \sum_{k=1}^{r_n}\frac{\xi_k(x)}{3^k}.$$

Then,

$$x - x_n = \sum_{k=r_n+1}^{\infty}\frac{\xi_k(x)}{3^k} \geq \frac{1}{3^{r_n+1}} > 0$$

and we have the following evaluation:

$$\begin{aligned}
\left|\frac{F_a(x)-F_a(x_n)}{x-x_n}\right| &= \frac{\left|\sum_{k=r_n+1}^{\infty}\Psi_k(x)\right|}{\sum_{k=r_n+1}^{\infty}\xi_k(x)/3^k}\\
&\geq \frac{\left|\sum_{k=r_n+1}^{\infty}\Psi_k(x)\right|}{1/3^{r_n}}\\
&= 3^{r_n}\prod_{l=1}^{r_n}\left|p\Big(\xi_l(x)\Big)\right|\cdot\left|q\Big(\xi_{r_n+1}(x)\Big)\right.\\
&\qquad\left.+\sum_{k=r_n+2}^{\infty}\prod_{l=r_n+1}^{k-1}p\Big(\xi_l(x)\Big)\cdot q\Big(\xi_k(x)\Big)\right|\\
&\geq 3^{r_n}\prod_{l=1}^{r_n}\left|p\Big(\xi_l(x)\Big)\right|\Bigg(q\Big(\xi_{r_n+1}(x)\Big)\\
&\qquad-\sum_{k=r_n+2}^{\infty}\prod_{l=r_n+1}^{k-1}\left|p\Big(\xi_l(x)\Big)\right|q\Big(\xi_k(x)\Big)\Bigg)\\
&\geq 3^{r_n}\prod_{l=1}^{r_n}\left|p\Big(\xi_l(x)\Big)\right|\\
&\qquad\left(q(1)-\sum_{k=1}^{\infty}\left|p(1)\right|\max_{0\leq l\leq 2}\left|p(l)\right|^{k-1}\max_{0\leq l\leq 2}q(l)\right)\\
&= 3^{r_n}\prod_{l=1}^{r_n}\left|p\Big(\xi_l(x)\Big)\right|\left(a-(2a-1)\sum_{k=1}^{\infty}a^k\right)\\
&= \frac{a(2-3a)}{1-a}\exp\left(\sum_{l=1}^{r_n}\log\left|3p\Big(\xi_l(x)\Big)\right|\right).
\end{aligned}$$

Using the following relations:

$$\begin{aligned}
\log\left|3p(0)\right| &= \log\left|3p(2)\right| = \log(3a),\\
\log\left|3p(1)\right| &= \log\left|3(1-2a)\right| = \log\left|\frac{27a^2-54a^3}{9a^2}\right|\\
&= \log\left|-\frac{1}{9a^2}\right| = -2\log(3a),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\left|\frac{F_a(x)-F_a(x_n)}{x-x_n}\right|\\
&\quad\geq \frac{a(2-3a)}{1-a}\exp\left(\log(3a)\sum_{l=1}^{r_n}c\Big(\xi_l(x)\Big)\right)\\
&\quad\geq \frac{a(2-3a)}{1-a}(3a)^{\sqrt{r_n}}.
\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\left|\frac{F_a(x)-F_a(x_n)}{x-x_n}\right| = \infty.$$

Namely, $F_a(x)$ is non-differentiable at x. From the previous lemma, we know that

$$\limsup_{n\to\infty}\frac{T_n(x)}{\sqrt{n}} \geq 1$$

holds almost everywhere in $[0,1)$, and so, we can conclude that $F_a(x)$ is non-differentiable almost everywhere in $[0,1)$.

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