Corwin–Greenleaf multiplicity functions for Hermitian symmetric spaces

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Abstract: Kobayashi's multiplicity-free theorem asserts that irreducible unitary highest weight representations π are multiplicity-free when restricted to any symmetric pairs if π is of scalar type. The Hua–Kostant–Schmid–Kobayashi branching laws embody this abstract theorem with explicit irreducible decomposition formulas of holomorphic discrete series representations with respect to symmetric pairs. In this paper, we study the 'classical limit' (geometry of coadjoint orbits) of a special case of these representation theoretic theorems in the spirit of the Kirillov–Kostant–Duflo orbit method.

First, we consider the Corwin–Greenleaf multiplicity function $n(\mathcal{O}^G, \mathcal{O}^K)$ for Hermitian symmetric spaces G/K. The first main theorem is that $n(\mathcal{O}^G, \mathcal{O}^K) \leq 1$ for any G-coadjoint orbit $\mathcal{O}^G$ and any K-coadjoint orbit $\mathcal{O}^K$ if $\mathcal{O}^G \cap \sqrt{-1}([\mathfrak{k}, \mathfrak{k}] + \mathfrak{p})^\perp \neq \emptyset$. Here, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. The second main theorem is a necessary and sufficient condition for $n(\mathcal{O}^G, \mathcal{O}^K) \neq 0$ by means of strongly orthogonal roots.

Key words: Hermitian symmetric space; Corwin–Greenleaf multiplicity function; orbit method; Kobayashi's multiplicity-free theorem; highest weight representations; branching law.

1. Corwin–Greenleaf multiplicity function. The orbit method pioneered by Kirillov and Kostant seeks to understand irreducible unitary representation by analogy with "quantization" procedures in mechanics. Physically, the idea of quantization is to replace a classical mechanical model (a phase space modelled by a symplectic manifold M) with a quantum mechanical model (a state space modelled by a Hilbert space $\mathcal{H}$) of the same system. The natural quantum analogue of the action of a group G on M by symplectomorphisms is a unitary representation of G on $\mathcal{H}$.

For a Lie group G, coadjoint orbits are symplectic manifolds, and the philosophy of the orbit method says that there should be a method of "quantization" to pass from coadjoint orbits for G to irreducible unitary representations of G. Kirillov proved that this works perfectly for nilpotent Lie groups, but many specialists have pointed out that the orbit method does not work very well for semisimple Lie groups [7,10,18,22]. However, we can still expect an intimate relation between the unitary dual of G and the set of (integral) coadjoint orbits even for a semisimple Lie group. We also expect that the orbit method should be "functorial", in the sense that branching laws of the restrictions (the side of unitary representation theory) are explained from the projection of coadjoint orbits (the side of symplectic geometry) and vise versa. Our present work is to establish such functorial property in a special setting, where multiplicity-free theorems of branching laws are known to be true.

Now, let us focus on the main theme of this paper. One of the fundamental problems in representation theory is to decompose a given representation into irreducibles [10]. Branching laws are one of the most important cases. Here, by *branching laws* we mean the irreducible decomposition in terms of a direct integral of an irreducible unitary representation π of a group G when restricted to a subgroup H:

$$\pi|_H \simeq \int_{\widehat{H}}^{\oplus} m_\pi(\nu)\nu d\mu(\nu). \tag{1.1}$$

Such a decomposition is unique, for example, if H is reductive or nilpotent, and the *multiplicity* m_π :

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$\widehat{H} \to \mathbf{N} \cup \{\infty\}$ makes sense as a measurable function on the unitary dual $\widehat{H}$. Here are two basic questions on multiplicities:

Problem 1.1.

a) *For which* (G, H, π), *the restriction* $\pi|_H$ *is multiplicity-free?*

b) *Relate quantum and classical pictures in the spirit of Kirillov–Kostant orbit method.*

As for (a), T. Kobayashi recently established a unified theory on multiplicity-free theorems for branching laws for both finite and infinite dimensional representations in a broad setting [12,16]. His theorem gives a uniform explanation for many known cases of multiplicity free results (e.g. Clebsch–Gordan formula, Pieri's rule, Plancherel formula for Riemannian symmetric spaces, $\mathrm{GL}_m - \mathrm{GL}_n$ duality, the Hua–Kostant–Schmid–Kobayashi formula [9], ...) and also presents many new cases of multiplicity free branching laws.

As for (b), it is well-known that the orbit method works perfectly for nilpotent Lie groups [7], but only partially for reductive groups.

For simply connected nilpotent Lie group G, building on the Kirillov isomorphism

$$\sqrt{-1}\,\mathfrak{g}^*/G \simeq \widehat{G},$$

Corwin and Greenleaf introduced the function $n(\mathcal{O}^G, \mathcal{O}^H)$. For later purpose, we review its definition in the general setting where G is a Lie group and H is its subgroup. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras, and write $\mathrm{pr} : \mathfrak{g}^* \to \mathfrak{h}^*$ for the natural projection. For coadjoint orbits $\mathcal{O}^G \subset \sqrt{-1}\mathfrak{g}^*$ and $\mathcal{O}^H \subset \sqrt{-1}\mathfrak{h}^*$, the *Corwin–Greenleaf multiplicity function* $n(\mathcal{O}^G, \mathcal{O}^H)$ is the number of H-orbits in the intersection $\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H)$. If π is attached to $\mathcal{O}^G$ and τ is attached to $\mathcal{O}^H$, then one expects that $n(\mathcal{O}^G, \mathcal{O}^H)$ coincides with $m_\pi(\nu)$. Research in this direction has been made extensively for nilpotent Lie groups and for certain solvable groups by Kirillov, Corwin, Greenleaf, Lipsman, and Fujiwara among others [1,3,4,20]. However, branching problems for semisimple Lie groups are hard in general, and substantial progress has been made only in the last decade or so [8]. In this paper, inspired by recent multiplicity-free theorems for semisimple Lie groups [9,11–13,16], and their underlying complex geometry [14,15], we seek for the counterpart in the orbit geometry associated to Hermitian symmetric spaces by means of the Corwin–Greenleaf multiplicity functions.

2. Main results. Suppose G is a non-compact simple Lie group, θ a Cartan involution of G, and $K := \{g \in G | \theta g = g\}$ a maximal compact subgroup. We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, corresponding to the Cartan involution θ. The group G is said to be of *Hermitian type*, if the Riemannian symmetric space G/K is a Hermitian symmetric space, or equivalently, if the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$ is non-trivial. The classification of simple Lie algebras $\mathfrak{g}$ of Hermitian type is given as follows:

$$\mathfrak{su}(p,q),\ \mathfrak{sp}(n,\mathbf{R}),\ \mathfrak{so}(m,2)\ (m \neq 2),\ \mathfrak{so}^*(2n),\ \mathfrak{e}_{6(-14)},\ \mathfrak{e}_{7(-25)}.$$

An irreducible representation π of G is a *highest weight representation of scalar type* if π is realized in the space of holomorphic sections for a G-equivariant holomorphic line bundle over the Hermitian symmetric space G/K. A typical example is a holomorphic discrete series (of scalar type), which means that π is realized as square-integrable holomorphic sections. For a Hermitian Lie group G, Harish-Chandra constructed infinitely many holomorphic discrete series representations of scalar type among others. Kobayashi's multiplicity-free theorem ([9], see also [13, Theorem A]) says:

Fact 2.1. *For any irreducible unitary highest weight representation* π *of scalar type of* G *and for any symmetric pair* (G, H), *the restriction* $\pi|_H$ *is multiplicity-free.*

On the other hand, a holomorphic discrete series representation π of scalar type may be regarded as a geometric quantization of a coadjoint orbit $\mathcal{O}^G$ in $\sqrt{-1}\mathfrak{g}^*$ satisfying

$$\mathcal{O}^G \cap \sqrt{-1}([\mathfrak{k},\mathfrak{k}] + \mathfrak{p})^\perp \neq \emptyset \tag{2.1}$$

(see [10, Example 2.7]). We note that $([\mathfrak{k},\mathfrak{k}] + \mathfrak{p})^\perp \neq \{0\}$ if and only if G is of Hermitian type. In the late 1990s (see also [17]), Kobayashi proposed the following conjecture as a counterpart of Fact 2.1:

Conjecture 2.2. *Let* $\mathcal{O}^G$ *be a* G*-coadjoint orbit satisfying* (2.1), *and* (G, H) *a symmetric pair. Then* $n(\mathcal{O}^G, \mathcal{O}^H) \leq 1$ *for any* H*-coadjoint orbit* $\mathcal{O}^H$ *in* $\sqrt{-1}\mathfrak{h}^*$.

The first main result of this paper is to give an affirmative solution to Conjecture 2.2 for $H = K$, namely, for a Riemannian symmetric pair (G, K):

Theorem A. $n(\mathcal{O}^G, \mathcal{O}^K) \leq 1$ for any K-coadjoint orbit $\mathcal{O}^K$ if $\mathcal{O}^G$ satisfies (2.1).

Since K is connected, an immediate corollary is the following topological result:

Corollary B. The intersection $\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^K)$ is connected for any coadjoint orbit $\mathcal{O}^G$ satisfying (2.1) and for any coadjoint orbit $\mathcal{O}^K$ in $\sqrt{-1}\mathfrak{k}^*$.

We note that the Blattner formula is just the identity of the characters and that its actural computation contains too many cancellations of positive and negative terms. This fact prevents us to tell explicitly whether the K-multiplicity is free or not in general. Likewise, the existing results on the 'classic limit' of the Blatter formula (e.g. [2]) does not yield directly Theorem A and Corollary B, to the best of our knowledge.

The second main result of this paper is a concrete criterion for $n(\mathcal{O}^G, \mathcal{O}^K) \neq 0$, equivalently, $\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^K) \neq \emptyset$. In order to state the theorem, let $\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} + \mathfrak{p}_+ + \mathfrak{p}_-$ be the irreducible decomposition as K-modules. We take a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{k}$, and fix a positive system $\Delta_c^+ := \Delta^+(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$. We write $\Delta_n^+ := \Delta^+(\mathfrak{p}_+, \mathfrak{t}_{\mathbf{C}})$ and set $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}}) := \Delta_c^+ \cup \Delta_n^+$.

Let $\{\gamma_1, \cdots, \gamma_r\}$ be the maximal set of strongly orthogonal roots [19] in Δ_n^+ satisfying:

(2.2) γ_j is the highest among $\{\alpha \in \Delta_n^+ : \alpha$ is strongly orthogonal to $\gamma_1, \cdots, \gamma_{j-1}\}$

for $1 \leq j \leq r$. By using the Killing form, we regard

$$\sqrt{-1}\mathfrak{c}(\mathfrak{k})^* \subset \sqrt{-1}\mathfrak{t}^* \subset \sqrt{-1}\mathfrak{k}^* \subset \sqrt{-1}\mathfrak{g}^*$$

corresponding to the inclusion $\mathfrak{c}(\mathfrak{k}) \subset \mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$. Then any K-coadjoint orbit $\mathcal{O}^K$ in $\sqrt{-1}\mathfrak{k}^*$ meets at a single point, say μ, in the dominant Weyl chamber. Then we write $\mathcal{O}_\mu^K$ for $\mathcal{O}^K$. Similarly, any coadjoint orbit $\mathcal{O}^G$ satisfying (2.1) meets at a single point, say λ, in $\sqrt{-1}\mathfrak{c}(\mathfrak{k})^*$. We write $\mathcal{O}_\lambda^G$ for this coadjoint orbit $\mathcal{O}^G$. We note that $\langle \lambda, \beta \rangle$ takes the same value for any $\beta \in \Delta_n^+$.

We are ready to state our second main result:

Theorem C. Supose $\langle \lambda, \beta \rangle > 0$ for any $\beta \in \Delta_n^+$. Then, $n(\mathcal{O}_\lambda^G, \mathcal{O}_\mu^K) \neq 0$ if and only if

$$\mu \in \lambda + \sum_{\substack{a_1 \geq \cdots \geq a_r \geq 0 \\ a_1, \cdots, a_r \in \mathbf{R}}} a_j \gamma_j.$$

A similar result holds in the case $\langle \lambda, \beta \rangle < 0$ for any $\beta \in \Delta_n^+$.

The branching law of scalar holomorphic discrete series representations π_λ for symmetric pairs is known as the Hua–Kostant–Schmid–Kobayashi formula. For the compact H, L.-K. Hua [5] found the formula but for $\mathfrak{e}_{6(-14)}$ and $\mathfrak{e}_{7(-25)}$, and later B. Kostant, W. Schmid [21] and some others gave a proof including the two exeptional cases. For general H, special cases were studied by H. Jakobsen, M. Vergne [6], J. Xie [23], etc., and the final form was established by T. Kobayashi [9] (see also [13, Theprem 8.3]). The formula for compact H amounts to:

$$\pi_\lambda^G|_K \simeq \sum_{\substack{a_1 \geq \cdots \geq a_r \geq 0 \\ a_1, \cdots, a_r \in \mathbf{N}}}^{\oplus} \pi^K_{\lambda + \sum_{j=1}^{l} a_j \gamma_j}.$$

Here, π_μ^K denotes the finite dimensional representation of K with highest weight μ if μ is a dominant integral weight and π_λ^G is an irreducible unitary lowest weight representation with minimal K-type π_λ^K. Thus Theorem C matches exactly this formula, and can be restated as:

Theorem C′. $n(\mathcal{O}_\lambda^G, \mathcal{O}_\mu^K) \neq 0$ if and only if

$$\mu \in \text{Convex-hull of } \mathrm{Supp}_K(\pi_\lambda^G|_K)$$

where $\mathcal{O}_\lambda^G \leftrightarrow \pi_\lambda^G$ and $\mathcal{O}_\mu^K \leftrightarrow \pi_\mu^K$ are the correspondences between coadjoint orbits and lowest weight modules.

As we formulated, the extension to the noncompact H case makes sense, and we hope to be back to this problem in a subsequent paper.

Sketch of the proof main results. The proof of Theorems A and C is based on the following steps:
Step 1) $\mathfrak{sl}(2, \mathbf{C})$-reduction by using the maximal set $\{\gamma_1, \cdots, \gamma_r\}$ of strongly orthogonal roots.
Step 2) Find a dominant chamber A_+ compatible with Δ_c^+.
Step 3) Find an explicit formula of $\mathrm{pr}(\mathrm{Ad}(a)Z)$. Here, Z is a normalized generator of $\sqrt{-1}\mathfrak{c}(\mathfrak{k})$.
Details proofs will appear elsewhere.

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