The tropical resultant

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Abstract: The resultant of two tropical polynomials satisfies the similar properties to the resultant of two polynomials over a field.

Key words: Tropical geometry; tropical semiring; max-plus algebra.

1. Preliminaries. We introduce some general theory on tropical geometry. For further details, please refer to [2–4].

Let $\mathbf{T}$ denote the set $\mathbf{R} \cup \{-\infty\}$. $\oplus$ and $\otimes$ are the tropical operators defined over $\mathbf{T}$ by $a \oplus b := \max(a, b), a \otimes b := a + b$. $(\mathbf{T}, \oplus, \otimes)$ is a semifield called *the tropical semifield.*

By $\mathbf{T}[\underline{x}] := \mathbf{T}[x_1, \ldots, x_n]$ we mean the set of tropical polynomials in n variables over the tropical semifield. For instance, $x^2 \oplus 0 = x^2 \oplus (-\infty)x \oplus 0$. We denote the set of tropical polynomial functions in n variables over the tropical semifield as $\mathrm{Poly}(\mathbf{T}^n)$*); $\mathrm{Poly}(\mathbf{T}^n) := \mathbf{T}[\underline{x}]/\sim$, where

$$F \sim G \Longleftrightarrow F(p) = G(p) \text{ for every } p \in \mathbf{T}^n.$$

Theorem 1.1 [3,5]. *Every nonconstant element of* $\mathrm{Poly}(\mathbf{T})$ *can be decomposed into the product of linear functions.*

In particular, $\mathbf{T}$ is "algebraically closed".

For a polynomial

$$F = \sum_{I \in \mathbf{Z}^n_{\geq 0}} a_I \underline{x}^I \in \mathbf{T}[\underline{x}],$$

we define the tropical hypersurface $V(F)$ as

$$V(F) = \{p = (p_1, \ldots, p_n) \in \mathbf{T}^n \mid F(p) = a_J p^J = a_{J'} p^{J'},\ {}^{\exists}J, J',\ J \neq J'\}.$$

Note that $V(F) \supset \{p \in \mathbf{T}^n \mid F(p) = -\infty\}$.

If both $F, G \in \mathbf{T}[\underline{x}]$ are the representatives of $f \in \mathrm{Poly}(\mathbf{T}^n)$, then $V(F) = V(G)$ holds. So we define $V(f)$ to be $V(F)$.

We call a point in $V(F)$ a zero of F. Following [1], we say that $F \in \mathbf{T}[\underline{x}]$ (resp. $f \in \mathrm{Poly}(\mathbf{T}^n)$) is tropically singular at $p \in \mathbf{T}^n$ if p is a zero of F (resp. f).

Let the determinant of a matrix $A \in M(n, \mathbf{T})$ to be defined as *the permanent* under tropical operators;

$$\det(A) := \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

2. Results. First, we define the tropical Sylvester matrix and the tropical resultant to a tropical polynomial as a natural analogy of ordinary ones.

For positive integers n, m, the tropical Sylvester matrix in $(n + m + 2)$ indeterminants $M((\zeta_0, \ldots, \zeta_n), (\eta_0, \ldots, \eta_m))$ is defined as

$$M((\zeta_0, \ldots, \zeta_n), (\eta_0, \ldots, \eta_m)) :=$$
$$\begin{pmatrix} \zeta_0 & \zeta_1 & \cdots\cdots & \zeta_n & & -\infty \\ & \zeta_0 & \zeta_1 & \cdots\cdots & \zeta_n & \\ & & \ddots & & & \ddots \\ -\infty & & \zeta_0 & \zeta_1 & \cdots\cdots & \zeta_n \\ \eta_0 & \eta_1 & \cdots\cdots & \eta_m & & -\infty \\ & \eta_0 & \eta_1 & \cdots\cdots & \eta_m & \\ & & \ddots & & & \ddots \\ -\infty & & \eta_0 & \eta_1 & \cdots\cdots & \eta_m \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \end{matrix}.$$

We define the tropical resultant $R((\zeta_0, \ldots, \zeta_n), (\eta_0, \ldots, \eta_m))$ as the determinant of the tropical Sylvester matrix;

$$R((\zeta_0, \ldots, \zeta_n), (\eta_0, \ldots, \eta_m)) := \det M((\zeta_0, \ldots, \zeta_n), (\eta_0, \ldots, \eta_m)).$$

Note that $R((\zeta_0, \ldots, \zeta_n), (\eta_0, \ldots, \eta_m))$ is an element of $\mathbf{T}[\zeta_0, \ldots, \zeta_n, \eta_0, \ldots, \eta_m]$.

For tropical polynomials $F = a_0 x^n \oplus \cdots \oplus a_n$,

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*) Usually $\mathbf{T}[\underline{x}]$ is confusingly used both for the polynomial semiring and $\mathrm{Poly}(\mathbf{T}^n)$.

$G = b_0x^m \oplus \cdots \oplus b_m$ $(n, m \geq 1)$, we denote $M((a_0, \ldots, a_n), (b_0, \ldots, b_m))$ as $M(F, G)$ and $R((a_0, \ldots, a_n), (b_0, \ldots, b_m))$ as $R(F, G)$.

Put $\Delta \in \mathbf{T}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ as

$$\Delta := \prod_{\substack{1\leq i\leq n\\1\leq j\leq m}} (x_i \oplus y_j).$$

Theorem 2.1. *For two nonconstant tropical polynomials*

$$F \sim a_0(x \oplus \alpha_1) \ldots (x \oplus \alpha_n)$$
$$G \sim b_0(x \oplus \beta_1) \ldots (x \oplus \beta_m)$$

the following holds:

- $R(F, G) = a_0^m b_0^n \Delta(\underline{\alpha}, \underline{\beta})$, *where* $\underline{\alpha} = (\alpha_1 \ldots, \alpha_n)$, $\underline{\beta} = (\beta_1, \ldots, \beta_m)$.
- $R(\ ,\ )$ *is tropically singular at* (F, G) *if and only if* $\Delta(\ ,\ )$ *is tropically singular at* $(\underline{\alpha}, \underline{\beta})$.

Remark 1. $\Delta(\ ,\ )$ is tropically singular at $(\underline{\alpha}, \underline{\beta})$ if and only if $\alpha_p = \beta_q$ holds for some p, q.

Corollary 2.2. *If $F \sim F'$ and $G \sim G'$ then $R(F, G) = R(F', G')$ holds and $R(\ ,\ )$ is tropically singular at (F, G) if and only if it is tropically singular at (F', G').*

This corollary shows that the tropical resultant can be naturally defined over tropical polynomial functions even though it is determined by the coefficients of tropical polynomials.

Definition 2. For two tropical polynomial functions $f, g \in \mathrm{Poly}(\mathbf{T})$ with the representatives being $F, G \in \mathbf{T}[x]$, we define $R(f, g)$ as $R(F, G)$. We say that $R(\ ,\ )$ is tropically singular at (f, g) if it is tropically singular at (F, G).

Main Theorem. *For two nonconstant tropical polynomial functions*

$$f = a_0(x \oplus \alpha_1) \ldots (x \oplus \alpha_n)$$
$$g = b_0(x \oplus \beta_1) \ldots (x \oplus \beta_m)$$

the following holds:

- $R(f, g) = a_0^m b_0^n \Delta(\underline{\alpha}, \underline{\beta})$, *where* $\underline{\alpha} = (\alpha_1 \ldots, \alpha_n)$, $\underline{\beta} = (\beta_1, \ldots, \beta_m)$.
- $R(\ ,\ )$ *is tropically singular at* (f, g) *if and only if* $\Delta(\ ,\ )$ *is tropically singular at* $(\underline{\alpha}, \underline{\beta})$.

Thus, in this sense, $R(F, G)$ equals $\Delta(\underline{\alpha}, \underline{\beta})$ including the singularity. In particular two tropical polynomial functions f, g have the same "zero" if and only if the resultant is tropically singular at (f, g).

3. Proof of Theorem 2.1.

In the rest of this paper, we shall prove Theorem 2.1.

Put $F = a_0x^n \oplus \cdots \oplus a_n$, $G = b_0x^m \oplus \cdots \oplus b_m$. Without loss of generality, we assume

- $a_0 = b_0 = 0$,
- $\alpha_1 = \alpha_2 = \cdots = \alpha_s > \alpha_{s+1} \geq \cdots \geq \alpha_n$,
- $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_m$,
- $\alpha_1 \geq \beta_1$.

Lemma 3.1. *If $R(F, G) = -\infty$, then $\Delta(\underline{\alpha}, \underline{\beta}) = -\infty$. The inverse also holds.*

Proof. Since $R(F, G) \geq \alpha_n^m \oplus \beta_m^n$, $R(F, G) = -\infty$ yields $\alpha_n = \beta_m = -\infty$. Then we have $\Delta(\underline{\alpha}, \underline{\beta}) = -\infty$.

On the other hand, if $\alpha_n = \beta_m = -\infty$ then every element of the $(m + n)$-th column of $M(F, G)$ is $-\infty$. So we have $R(F, G) = -\infty$. □

We assume $R(F, G) \neq -\infty$ from here on (and $\Delta(\underline{\alpha}, \underline{\beta}) \neq -\infty$ also). Then $\beta_1 > -\infty$.

Put $M_1(F, G)$ (resp. $M_2(F, G)$) to be the submatrix obtained by deleting the first row (resp. $(m + 1)$-th row) and the first column of $M(F, G)$. Then $R(F, G) = a_0 \det M_1(F, G) \oplus b_0 \det M_2(F, G)$.

Set the degree of (F, G) as

$$\deg(F, G) = \begin{cases} 1, & \text{if } n \text{ or } m = 1, \\ n + m, & \text{otherwise.} \end{cases}$$

We will show the theorem inductively over $\deg(F, G)$.

The following lemma is obvious from the direct calculation.

Lemma 3.2. *If n or $m = 1$, then the theorem holds.*

We assume both $n, m \geq 2$ from here on.

3.1. The equality.

Put $\widetilde{a_i} := \alpha_1 \ldots \alpha_i$, $\widetilde{b_j} = \beta_1 \ldots \beta_j$ $(1 \leq i \leq n, 1 \leq j \leq m)$. Then

$$F \sim \widetilde{F} := x^n \oplus \widetilde{a_1}x^{n-1} \oplus \cdots \oplus \widetilde{a_n},$$
$$G \sim \widetilde{G} := x^m \oplus \widetilde{b_1}x^{m-1} \oplus \cdots \oplus \widetilde{b_m}.$$

$\deg(F, G) = \deg(\widetilde{F}, \widetilde{G})$ holds.

The following lemma holds since $\alpha_1 \geq \alpha_i$ holds for every i.

Lemma 3.3. *$\widetilde{a_i} \leq \widetilde{a_1}\widetilde{a_{i-1}}$ holds for every i.*

Denote the (i, j)-th element of a matrix A by A_{ij}.

Lemma 3.4. *There exists $\sigma \in \mathfrak{S}_{n+m-1}$ such that $\det M_2(\widetilde{F}, \widetilde{G}) = \prod_i M_2(\widetilde{F}, \widetilde{G})_{i,\sigma(i)}$ and $i - \sigma(i) \neq 1$ for every $i \leq m$.*

Proof. For an arbitary σ, put $N(\sigma) = \#\{i \leq m \mid i - \sigma(i) = 1\}$. Put S to be the subset of $\mathfrak{S}_{n+m-1}$ defined by $S = \{\sigma \mid \det M_2(\widetilde{F}, \widetilde{G}) = \prod_i M_2(\widetilde{F}, \widetilde{G})_{i,\sigma(i)}\}$.

Suppose $\sigma_0 \in S$ satisfies $N(\sigma_0) \leq N(\sigma)$ for every $\sigma \in S$. If $N(\sigma_0) \neq 0$, then put $t = \min\{i \leq m \mid$

$i - \sigma(i) = 1\}$ and set $\sigma_1 = \sigma \circ (t, t-1)$. Then σ_1 is an element of S satisfying $N(\sigma_1) < N(\sigma_0)$ from Lemma 3.3. □

Similary, we have the following lemma.

Lemma 3.5. *There exists* $\sigma \in \mathfrak{S}_{n+m-1}$ *such that* $\det M_1(\widetilde{F}, \widetilde{G}) = \prod_i M_1(\widetilde{F}, \widetilde{G})_{i,\sigma(i)}$ *and* $i - \sigma(i) \neq m$.

Proposition 3.6. $R(\widetilde{F}, \widetilde{G}) = \Delta(\underline{\alpha}, \underline{\beta})$.

Proof. From Lemma 3.4, we have $\det M_2(\widetilde{F}, \widetilde{G}) = R(\widetilde{F'}, \widetilde{G})$, where

$$\widetilde{F'} = \widetilde{a_1} x^{n-1} \oplus \cdots \oplus \widetilde{a_n} = \alpha_1 (x \oplus \alpha_2) \ldots (x \oplus \alpha_n).$$

Since $\deg(\widetilde{F'}, \widetilde{G}) < \deg(\widetilde{F}, \widetilde{G})$, we have $\det M_2(\widetilde{F}, \widetilde{G}) = \alpha_1^m \prod_{i \neq 1} (\alpha_i \oplus \beta_j)$ by the assumption of the induction.

Similarly, we have $\det M_1(\widetilde{F}, \widetilde{G}) = \beta_1^n \prod_{j \neq 1} (\alpha_i \oplus \beta_j)$ and thus

$$\begin{aligned} R(\widetilde{F}, \widetilde{G}) &= \beta_1^n \prod_{j \neq 1} (\alpha_i \oplus \beta_j) \oplus \alpha_1^m \prod_{i \neq 1} (\alpha_i \oplus \beta_j) \\ &= \Delta(\underline{\alpha}, \underline{\beta}). \end{aligned}$$

□

We will now show that $R(\widetilde{F}, \widetilde{G}) = R(F, G)$.

Lemma 3.7. $a_i \leq \widetilde{a_i}$, $b_j \leq \widetilde{b_j}$ *for every* i, j.

Proof. Suppose $a_p > \widetilde{a_p}$ for some p. Then

$$\begin{aligned} \widetilde{F}(\alpha_p) &= \alpha_p^n \oplus \alpha_1 \alpha_p^{n-1} \oplus \cdots \oplus \alpha_1 \ldots \alpha_n \\ &= \alpha_1 \ldots \alpha_{p-1} \alpha_p^{n-p+1} \\ &= \widetilde{a_p} \alpha_p^{n-p} \\ &< a_p \alpha_p^{n-i} \leq F(\alpha_p). \end{aligned}$$

□

Remark 3. This lemma can also be shown from the general theory of tropical geometry since their extended Newton polytopes coincide.

Corollary 3.8. $R(F, G) \leq R(\widetilde{F}, \widetilde{G})$.

If $a_1 = -\infty$, we have $R(F, G) = R(x^n \oplus \epsilon x^{n-1} \oplus a_2 x^{n-2} \oplus \cdots \oplus a_n, G)$ for sufficiently small $\epsilon > -\infty$ since $R(F, G) \neq -\infty$. So we assume $a_1 \neq -\infty$ from here on.

Let the integers $\alpha'_2 \geq \cdots \geq \alpha'_n$ satisfy

$$a_1 x^{n-1} \oplus \cdots \oplus a_n \sim a_1 (x \oplus \alpha'_2) \ldots (x \oplus \alpha'_n).$$

Lemma 3.9. $\alpha'_2, \ldots, \alpha'_n$ *satisfies the followings*

$$\begin{cases} \alpha_i \leq \alpha'_i, & \text{if } 2 \leq i \leq s, \\ \alpha_i = \alpha'_i, & \text{if } i > s, \end{cases}$$
$$\alpha_1 \ldots \alpha_s = a_1 \alpha'_2 \ldots \alpha'_s.$$

Proof. Since $x^n \oplus a_1 x^{n-1} \oplus \cdots \oplus a_n \sim (x \oplus \alpha_1) \ldots (x \oplus \alpha_n)$, we have

$$x^n \oplus a_1 x^{n-1} \oplus \cdots \oplus a_n |_{x \leq \alpha_1} = a_1 x^{n-1} \oplus \cdots \oplus a_n |_{x \leq \alpha_1}$$

as the tropical polynomial functions. Hence the first part follows.

Then since

$$\begin{aligned} \alpha_1^n &= a_1 \alpha_1^{n-1} \oplus \cdots \oplus a_n \\ &= a_1 (\alpha_1 \oplus \alpha'_2) \ldots (\alpha_1 \oplus \alpha'_n) \\ &= a_1 \alpha'_2 \ldots \alpha'_s \alpha_1^{n-s} \end{aligned}$$

and $\alpha_1 = \cdots = \alpha_s \neq -\infty$ holds, we have

$$\alpha_1 \ldots \alpha_s = a_1 \alpha'_2 \ldots \alpha'_s.$$

□

Proposition 3.10. $R(F, G) \geq R(\widetilde{F}, \widetilde{G})$.

Proof. Since $\det M_2(F, G) \geq R(a_1 x^{n-1} \oplus \cdots \oplus a_n, G)$ holds,

$$\begin{aligned} R(F, G) &\geq \det M_2(F, G) \\ &\geq R(a_1 x^{n-1} \oplus \cdots \oplus a_n, G) \\ &= a_1^m \prod_{i \neq 1} (\alpha'_i \oplus \beta_j) \\ &= a_1^m \alpha'^m_2 \ldots \alpha'^m_s \prod_{i > s} (\alpha'_i \oplus \beta_j) \\ &= \alpha_1^m \ldots \alpha_s^m \prod_{i > s} (\alpha_i \oplus \beta_j) \\ &= \prod (\alpha_i \oplus \beta_j) = R(\widetilde{F}, \widetilde{G}). \end{aligned}$$

□

So we have $R(\widetilde{F}, \widetilde{G}) \geq R(F, G) \geq R(\widetilde{F}, \widetilde{G})$.

3.2. Tropical singularity.

Lemma 3.11. *If* $\beta_1 = -\infty$, *then* $R(\ ,\ )$ *is tropically singular at* (F, G) *if and only if* $\Delta(\ ,\ )$ *is tropically singular at* $(\underline{\alpha}, \underline{\beta})$.

Proof. Obvious from calculation. If $a_n = -\infty$, then $F = x(a_0 x^{n-1} \oplus \cdots \oplus a_{n-1})$ so $\alpha_n = -\infty$. □

We assume β_1 (and α_1 also) not to be $-\infty$ from here on.

Note that $\det M_1(\ ,\ )$ and $\det M_2(\ ,\ )$ are the elements of $\mathbf{T}[\zeta_0, \ldots, \zeta_n, \eta_0, \ldots, \eta_m]$.

From Proposition 3.10, we have $R(F, G) = \det M_2(F, G)$ ($\geq \det M_1(F, G)$). Thus $R(\ ,\ )$ is tropically singular at (F, G) if and only if either $\det M_1(F, G) = \det M_2(F, G)$ or $\det M_2(\ ,\ )$ is tropically singular at (F, G). Suppose $\det M_1(F, G) = \det M_2(F, G)$ holds. Then since $\det M_1(F, G) \leq \det M_1(\widetilde{F}, \widetilde{G}) = \beta_1^n \prod_{j \neq 1} (\alpha_i \oplus \beta_j)$, we have $\alpha_1 = \beta_1$. On the other hand, if $\alpha_1 = \beta_1$ holds, then we have $\det M_1(F, G) = \det M_2(F, G)$ and thus $R(\ ,\ )$ is tropically singular at (F, G).

Suppose $\det M_2(F, G) > \det M_1(F, G)$, or equivalently, $\alpha_1 > \beta_1$. The following lemma is a stronger version of Lemma 3.4.

Lemma 3.12. *If* $\sigma \in \mathfrak{S}_{n+m-1}$ *satisfies* $\det M_2(\widetilde{F}, \widetilde{G}) = \prod_i M_2(\widetilde{F}, \widetilde{G})_{i,\sigma(i)}$, *then* $\sigma(i) - i \geq s - 1$ *holds for every* $i \leq m$.

Proof. As before, let S be the subset of $\mathfrak{S}_{n+m-1}$ defined by $S = \{\sigma \mid \det M_2(\widetilde{F}, \widetilde{G}) = \prod_i M_2(\widetilde{F}, \widetilde{G})_{i,\sigma(i)}\}$. For an arbitary σ, put $N'(\sigma) = \#\{i \leq m \mid \sigma(i) - i < s - 1\}$.

Suppose $\sigma_0 \in S$ satisfies $N'(\sigma_0) \geq 1$ and $N'(\sigma_0) = \min\{N'(\sigma) \mid \sigma \in S\}$. Put $r = \max\{\sigma_0(i) \mid \sigma_0(i) - i < s - 1,\ i \leq m\}$. Then either $\sigma_0^{-1}(r+1) < \sigma_0^{-1}(r)$ or $m + 1 \leq \sigma_0^{-1}(r+1) \leq m + r + 1$ holds. We define σ_1 as follows:

- *Case* 1: $\sigma_0^{-1}(r+1) < \sigma_0^{-1}(r)$;
  Put $\sigma_1 = \sigma_0 \circ (\sigma_0^{-1}(r), \sigma_0^{-1}(r+1))$.
- *Case* 2: $m + 1 \leq \sigma_0^{-1}(r+1) < m + r + 1$;
  Put $\sigma_1 = \sigma_0 \circ (\sigma_0^{-1}(r), \sigma_0^{-1}(r+1))$.
- *Case* 3: $\sigma_0^{-1}(r+1) = m + r + 1$;
  Put $\sigma_1 = \sigma_0 \circ (\sigma_0^{-1}(r), \sigma_0^{-1}(r+1) - 1, \sigma_0^{-1}(r+1))$.

Then σ_1 satisfies $\prod_i M_2(\widetilde{F}, \widetilde{G})_{i,\sigma_1(i)} > \prod_i M_2(\widetilde{F}, \widetilde{G})_{i,\sigma_0(i)}$. A contradiction. □

Corollary 3.13. $\det M_2(\ ,\ )$ *is tropically singular at* $(\widetilde{F}, \widetilde{G})$ *if and only if* $R(\ ,\ )$ *is tropically singular at* $(\widetilde{a_1}x^{n-1} \oplus \cdots \oplus \widetilde{a_n}, \widetilde{G})$.

Proof. $\Leftarrow$ is obvious since $\det M_2(\widetilde{F}, \widetilde{G}) = R(\widetilde{a_1}x^{n-1} \oplus \cdots \oplus \widetilde{a_n}, \widetilde{G})$. $\Rightarrow$ is a consequence of the previous lemma. □

Thus we have

$R(\ ,\ )$ is tropically singular at (F, G)
$\Downarrow$ (by assumption)
$\det M_2(\ ,\ )$ is tropically singular at (F, G)
$\Downarrow$ $(a_i \leq \widetilde{a_i}, b_j \leq \widetilde{b_j})$
$\det M_2(\ ,\ )$ is tropically singular at $(\widetilde{F}, \widetilde{G})$
$\Downarrow$ (Corollary 3.13)
$R(\ ,\ )$ is tropically singular at
$$(\widetilde{a_1}x^{n-1} \oplus \cdots \oplus \widetilde{a_n}x^{n-1}, \widetilde{G})$$
$\Downarrow$ (assmption of induction)
$\Delta(\ ,\ )$ is tropically singular at $(\underline{\alpha}, \underline{\beta})$

and

$\Delta(\ ,\ )$ is tropically singular at $(\underline{\alpha}, \underline{\beta})$
$\Downarrow$ $(\alpha_1 > \beta_1)$
$\alpha_p = \beta_q,\ {}^{\exists}p \geq 2, {}^{\exists}q$
$\Downarrow$ (Lemma 3.9)
$\alpha'_p = \beta_q,\ {}^{\exists}p \geq 2, {}^{\exists}q$
$\Downarrow$
$\Delta(\ ,\ )$ is tropically singular at
$$((\alpha'_2, \ldots, \alpha'_n), (\beta_1, \ldots, \beta_m))$$
$\Downarrow$ (assmption of induction)
$R(\ ,\ )$ is tropically singular at
$$(a_1x^{n-1} \oplus \cdots \oplus a_nx^{n-1}, G)$$
$\Downarrow$ $(R(a_1x^{n-1} \oplus \cdots \oplus a_n, G) = \det M_2(F, G))$
$\det M_2(\ ,\ )$ is tropically singular at (F, G)
$\Downarrow$
$R(\ ,\ )$ is tropically singular at (F, G).

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