Dedekind sums in finite characteristic

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(Communicated by Shigefumi MORI, M.J.A., June 10, 2008)

Abstract: This paper is concerned with Dedekind sums in finite characteristic. We introduce Dedekind sums for lattices, and establish the reciprocity law for them.

Key words: Dedekind sums; lattices; Drinfeld modules.

1. Introduction. This paper is a résumé of our results, and the details will be published elsewhere.

For two relatively prime integers $a, c \in \mathbf{Z}$ with $c \neq 0$, we define the classical Dedekind sum in the form

$$s(a,c) = \frac{1}{c} \sum_{k \in (\mathbf{Z}/c\mathbf{Z})-\{0\}} \cot\left(\pi \frac{k}{c}\right) \cot\left(\pi \frac{ak}{c}\right).$$

As is well known, $s(a,c)$ has the following properties:
(1) $s(-a,c) = -s(a,c)$.
(2) If $a \equiv a' \pmod{c}$, then $s(a,c) = s(a',c)$.
(3) (Reciprocity law) For two relatively prime integers $a, c \in \mathbf{Z} - \{0\}$,

$$s(a,c) + s(c,a) = \frac{1}{3}\left(\frac{a}{c} + \frac{1}{ac} + \frac{c}{a}\right) - \operatorname{sign}(ac).$$

The sum $s(a,c)$ is related to the module $\mathbf{Z}$. In [4], Sczech defined the Dedekind sum for a given lattice $\mathbf{Z}w_1 + \mathbf{Z}w_2$. Okada [3] introduced the Dedekind sum for a given function field. His Dedekind sum is related to the $\mathbf{F}_q[T]$-module L corresponding to the Carlitz module (cf. 2.1). Inspired by Okada's result, we defined in [2] the Dedekind sum for a given finite field. Our previous result is related to a given finite field itself. Observing these former results, we have noticed that it is possible to define the Dedekind sum for a given lattice in finite characteristic. In this paper, we introduce Dedekind sums for lattices, and establish the reciprocity law for them.

Our results is divided into two parts. Section 2 deals with function fields case. In section 3, we discuss finite fields case.

2000 Mathematics Subject Classification. Primary 11F20; Secondary 11T50, 11G09.

2. Function field Dedekind sums. In this section we use the following notations. Let $\mathbf{F}_q$ be the finite field with q elements, $A = \mathbf{F}_q[T]$ the ring of polynomials in an indeterminate T, $K = \mathbf{F}_q(T)$ the quotient field of A, $|\ |$ the normalized absolute value on K such that $|T| = q$, K_∞ the completion of K with respect to $|\ |$, $\overline{K_\infty}$ a fixed algebraic extension of K_∞, and C the completion of $\overline{K_\infty}$. We denote by $\sum'$, $\prod'$ the sum over non-zero elements, the product over non-zero elements, respectively.

2.1. *A*-lattices. A rank r *A-lattice* Λ in C means a finitely generated A-submodule of rank r in C that is discrete in the topology of C. For such an A-lattice Λ, define the Euler product

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda}{}' \left(1 - \frac{z}{\lambda}\right).$$

The product converges uniformly on bounded sets in C, and defines a map $e_\Lambda : C \to C$. The map e_Λ has the following properties:
• e_Λ is entire in the rigid analytic sense, and surjective;
• e_Λ is $\mathbf{F}_q$-linear and Λ-periodic;
• e_Λ has simple zeros at the points of Λ, and no other zeros;
• $de_\Lambda(z)/dz = e'_\Lambda(z) = 1$. Hence we have

$$\text{(2.1)} \qquad \frac{1}{e_\Lambda(z)} = \frac{e'_\Lambda(z)}{e_\Lambda(z)} = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda}.$$

An $\mathbf{F}_q$-linear ring homomorphism

$$\phi : A \to \operatorname{End}_C(\mathbf{G}_a), \quad a \mapsto \phi_a$$

is said to be a *Drinfeld module* of rank r over C if ϕ satisfies

$$\phi_T = T + a_1\tau + \cdots + a_r\tau^r, \quad a_r \neq 0$$

for some $a_i \in C$, where τ denotes the q-th power morphism in $\mathrm{End}_C(\mathbf{G}_a)$. Given a rank r A-lattice Λ, there exists a unique rank r Drinfeld module ϕ^Λ with the condition $e_\Lambda(az) = \phi_a^\Lambda(e_\Lambda(z))$ for all $a \in A$. The association $\Lambda \mapsto \phi^\Lambda$ yields a bijection of the set of A-lattices of rank r in C with the set of Drinfeld modules of rank r over C. The rank one Drinfeld module ρ defined by $\rho_T = T + \tau$ is said to be the *Carlitz module*. We denote the A-lattice associated to ρ by L.

We recall the Newton formula for power sums of the zeros of a polynomial.

Proposition 2.1 (The Newton formula cf. [1]). *Let*

$$f(X) = X^n + c_1X^{n-1} + \cdots + c_{n-1}X + c_n$$

be a polynomial, and $\alpha_1, \ldots, \alpha_n$ *the roots of* $f(X)$. *For each positive integer* k, *put*

$$T_k = \alpha_1^k + \cdots + \alpha_n^k.$$

Then

$$T_k + c_1T_{k-1} + \cdots + c_{k-1}T_1 + kc_k = 0 \quad (k \le n),$$
$$T_k + c_1T_{k-1} + \cdots + c_{n-1}T_{k-n+1} + c_nT_{k-n} = 0 \quad (k \ge n).$$

Using this formula, we have

Proposition 2.2. *Let* Λ *be a rank* r A*-lattice in* C, *and take a non-zero element* $a \in A$. *For* $m = 1, 2, \ldots, q-2$, *we have*

$$\frac{a^m}{e_\Lambda(az)^m} = \sum_{\lambda \in \Lambda/a\Lambda} \frac{1}{e_\Lambda(z - \lambda/a)^m}.$$

For any non-zero element $c \in A$, set

$$R(c) = \{e_\Lambda(\lambda/c) \mid \lambda \in \Lambda/c\Lambda\} - \{0\}.$$

In other words, $R(c)$ consists of the non-zero roots of $\phi_c(z)$. Let Λ be a rank r A-lattice in C corresponding to the Drinfeld module ϕ with

$$\phi_c(z) = \sum_{i=0}^{n} l_i(c) z^{q^i}, \tag{2.2}$$

where $n = r \deg c, l_n(c) \neq 0$, and $l_0(c) = c$.

Proposition 2.3.

$$\sum_{\alpha \in R(c)} \alpha^{-m} = \begin{cases} 0 & (m = 1, \ldots, q-2) \\ l_1(c)/c & (m = q-1) \end{cases}.$$

In particular, if $\phi = \rho$, *the Carlitz module, then*

$$\sum_{\alpha \in R(c)} \alpha^{-q+1} = \frac{c^{q-1} - 1}{T^q - T}.$$

2.2. Function field Dedekind sums. Observing that (2.1) is similar to a formula for $\pi \cot \pi z$, for an A-lattice Λ of finite rank in C, let us define Dedekind sum as follows:

Definition 2.4. Let $a, c \in A - \mathbf{F}_q$ be relatively prime elements. In other words, assume $Aa + Ac = A$. For $m = 1, \ldots, q-2$, define

$$s_m(a,c)_\Lambda = \frac{1}{c^m} \sum_{\lambda \in \Lambda/c\Lambda}{}' e_\Lambda\left(\frac{\lambda}{c}\right)^{-q+1+m} e_\Lambda\left(\frac{a\lambda}{c}\right)^{-m}.$$

Moreover, we define

$$s_0(c)_\Lambda = s_0(a,c)_\Lambda = \sum_{\lambda \in \Lambda/c\Lambda}{}' e_\Lambda\left(\frac{\lambda}{c}\right)^{-q+1}.$$

We call $s_m(a,c)_\Lambda$ the m-th *Dedekind-Drinfeld sum* for Λ. In particular, if L is the rank one A-lattice associated to the Carlitz module ρ, then $s_m(a,c)_L$ is called the m-th *Dedekind-Carlitz sum.*

Remark 2.5. (1) In [3], Okada defines the Dedekind-Carlitz sum. Our definition generalizes it. (2) It is possible to define Dedekind-Drinfeld sums in the same way for arbitrary function field instead of $K = \mathbf{F}_q(T)$.

It follows from Proposition 2.3 that

$$s_0(c)_\Lambda = s_0(a,c)_\Lambda = \frac{l_1(c)}{c},$$

where $l_1(c)$ is the coefficient of z^q in $\phi_c(z)$ as in (2.2). In particular, regarding the lattice L associated to the Carlitz module ρ,

$$s_0(c)_L = s_0(a,c)_L = \frac{c^{q-1} - 1}{T^q - T}.$$

The following result is analogous to the properties (1), (2) of the classical Dedekind sums in section one.

Proposition 2.6. *Dedekind sums* $s_m(a,c)_\Lambda$ $(m = 1, \ldots, q-2)$ *satisfy the following properties*:
(1) *For any* $\alpha \in \mathbf{F}_q^*$, $s_m(\alpha a, c)_\Lambda = \alpha^{-m} s_m(a,c)_\Lambda$.
(2) *If* $a, a' \in A$ *satisfy* $a - a' \in cA$, *then* $s_m(a,c)_\Lambda = s_m(a',c)_\Lambda$.
(3) *Take* $b \in A$ *with* $ab - 1 \in cA$. *Then* $s_m(b,c)_\Lambda = c^{q-1-2m} s_{q-1-m}(a,c)_\Lambda$.

2.3. Function field reciprocity law. We present the reciprocity law for our Dedekind sums. Let $a, c \in A - \mathbf{F}_q$ be relatively prime elements, and $m = 1, \ldots, q-2$.

Theorem 2.7 (Function field reciprocity law I).

$$s_m(a,c)_\Lambda + (-1)^{m-1} s_m(c,a)_\Lambda = \sum_{r=1}^{m-1} \frac{(-1)^{m-r} s_{m-r}(c,a)_\Lambda}{a^r c^r} \cdot \binom{m+1}{r} + \frac{s_0(c)_\Lambda + m \cdot s_0(a)_\Lambda}{a^m c^m}.$$

As a corollary to this result, the next theorem is obtained.

Theorem 2.8 (Function field reciprocity law II)**.**

$$s_m(a,c)_\Lambda + (-1)^{m-1} s_m(c,a)_\Lambda = \sum_{r=1}^{m-1} \frac{(-1)^{r-1}\Big(s_{m-r}(a,c)_\Lambda + (-1)^{m-1} s_{m-r}(c,a)_\Lambda\Big)\binom{m+1}{r}}{2a^r c^r} + \frac{\Big(m + (-1)^{m-1}\Big)\Big(s_0(a)_\Lambda + (-1)^{m-1} s_0(c)_\Lambda\Big)}{2a^m c^m}.$$

Example 2.9. Using the notation in the previous subsection, we have

$$s_1(a,c)_\Lambda + s_1(c,a)_\Lambda = \frac{a l_1(c) + c l_1(a)}{a^2 c^2},$$

$$s_3(a,c)_\Lambda + s_3(c,a)_\Lambda = \frac{2s_2(a,c)_\Lambda + 2s_2(c,a)_\Lambda}{ac} - \frac{a l_1(c) + c l_1(a)}{a^4 c^4}.$$

In particular, if $\Lambda = L$, then

$$s_1(a,c)_L + s_1(c,a)_L = \frac{a^{q-1} + c^{q-1} - 2}{ac(T^q - T)},$$

$$s_3(a,c)_L + s_3(c,a)_L = \frac{2s_2(a,c)_L + 2s_2(c,a)_L}{ac} - \frac{a^{q-1} + c^{q-1} - 2}{a^3 c^3 (T^q - T)}.$$

3. Finite field Dedekind sums. In this section, we use the following notations.

$K = \mathbf{F}_q$: the finite field with q elements.
$\overline{K}$: an algebraic closure of K.
$\sum'$: the sum over non-zero elements.
$\prod'$: the product over non-zero elements.

3.1. Lattices. A *lattice* Λ in $\overline{K}$ means a linear K-subspace in $\overline{K}$ of finite dimension. For such a lattice Λ, we define the Euler product

$$e_\Lambda(z) = z \prod_{\lambda\in\Lambda}{}' \left(1 - \frac{z}{\lambda}\right).$$

The product defines a map $e_\Lambda : \overline{K} \to \overline{K}$. The map e_Λ has the following properties:

• e_Λ is $\mathbf{F}_q$-linear and Λ-periodic.
• If $\dim_K \Lambda = r$, then $e_\Lambda(z)$ has the form

$$e_\Lambda(z) = \sum_{i=0}^{r} \alpha_i(\Lambda) z^{q^i}, \tag{3.1}$$

where $\alpha_0(\Lambda) = 1$ and $\alpha_r(\Lambda) \neq 0$.
• e_Λ has simple zeros at the points of Λ, and no other zeros.
• $de_\Lambda(z)/dz = e'_\Lambda(z) = 1$. Hence we have

$$\frac{1}{e_\Lambda(z)} = \frac{e'_\Lambda(z)}{e_\Lambda(z)} = \sum_{\lambda\in\Lambda} \frac{1}{z-\lambda}. \tag{3.2}$$

Using the Newton formula, we have

Proposition 3.1. *Let Λ be a lattice in $\overline{K}$, and take a non-zero element $a \in \overline{K}$. For $m = 1, 2, \ldots, q-2$, we have*

$$\frac{a^m}{e_\Lambda(az)^m} = \sum_{x\in\Lambda} \frac{1}{(z - x/a)^m}.$$

For $b \in \overline{K} - \{0\}$, set

$$R(b) = \{\lambda/b \mid \lambda \in \Lambda\} - \{0\}.$$

Lemma 3.2.

$$\sum_{x\in R(b)} x^{-m} = \begin{cases} 0 & (m = 1, \ldots, q-2) \\ \alpha_1(\Lambda) b^{q-1} & (m = q-1) \end{cases},$$

where $\alpha_1(\Lambda)$ is as in (3.1).

3.2. Finite field Dedekind sums. Observing that (3.2) is similar to a formula for $\pi \cot \pi z$, for a lattice Λ in $\overline{K}$, we define Dedekind sum as follows:

Definition 3.3. Set

$$\widetilde{\Lambda} = \{x \in \overline{K} \mid x\lambda \in \Lambda \text{ for some } \lambda \in \Lambda\}.$$

We choose $c, a \in \overline{K} - \{0\}$ such that $a/c \notin \widetilde{\Lambda}$. For $m = 1, \ldots, q-2$, define

$$s_m(a,c)_\Lambda = \frac{1}{c^m} \sum_{\lambda\in\Lambda}{}' \left(\frac{\lambda}{c}\right)^{-q+1+m} e_\Lambda\left(\frac{a\lambda}{c}\right)^{-m}.$$

Moreover, we define

$$s_0(c)_\Lambda = s_0(a,c)_\Lambda = \sum_{\lambda\in\Lambda}{}' \left(\frac{\lambda}{c}\right)^{-q+1}.$$

We call $s_m(a,c)_\Lambda$ the m-th *finite Dedekind sum* for Λ.

Remark 3.4. In [2], we defined the Dedekind sum for $\Lambda = K$. Our definition generalizes it.

It follows from Lemma 3.2 that

$$s_0(c)_\Lambda = s_0(a,c)_\Lambda = \alpha_1(\Lambda) c^{q-1},$$

where $\alpha_1(\Lambda)$ is the coefficient of z^q in $e_\Lambda(z)$ as in (3.1).

The following result is analogous to the properties (1), (2) of the classical Dedekind sums in section one.

Proposition 3.5. *Dedekind sums* $s_m(a,c)_\Lambda$ $(m = 1, \ldots, q-1)$ *satisfy the following properties*:
(1) *For any* $\alpha \in K^*$, $s_m(\alpha a, c)_\Lambda = \alpha^{-m} s_m(a,c)_\Lambda$.
(2) *If* $a, a' \in \overline{K}$ *satisfy* $a - a' \in c\Lambda$, *then* $s_m(a,c)_\Lambda = s_m(a',c)_\Lambda$.

3.3. Finite field reciprocity law. We present the reciprocity law for our Dedekind sums. Let a, c be the elements of $\overline{K} - \{0\}$ such that $a/c \notin \widetilde{\Lambda}$.

Theorem 3.6 (Finite field reciprocity law I)**.** *For* $m = 1, \ldots, q-2$, *we have*

$$s_m(a,c)_\Lambda + (-1)^{m-1} s_m(c,a)_\Lambda = \sum_{r=1}^{m-1} \frac{(-1)^{m-r} s_{m-r}(c,a)_\Lambda}{a^r c^r} \cdot \binom{m+1}{r} + \frac{s_0(c)_\Lambda + m \cdot s_0(a)_\Lambda}{a^m c^m}.$$

As a corollary to this result, the next theorem is obtained.

Theorem 3.7 (Finite field reciprocity law II)**.** *For* $m = 1, \ldots, q-2$, *we have*

$$s_m(a,c)_\Lambda + (-1)^{m-1} s_m(c,a)_\Lambda = \sum_{r=1}^{m-1} \frac{(-1)^{r-1}\Big(s_{m-r}(a,c)_\Lambda + (-1)^{m-1} s_{m-r}(c,a)_\Lambda\Big)\binom{m+1}{r}}{2a^r c^r} + \frac{\Big(m + (-1)^{m-1}\Big)\Big(s_0(a)_\Lambda + (-1)^{m-1} s_0(c)_\Lambda\Big)}{2a^m c^m}.$$

Example 3.8. Using the notation in the previous subsection, we have

$$s_1(a,c)_\Lambda + s_1(c,a)_\Lambda = \frac{\alpha_1(\Lambda)(a^{q-1} + c^{q-1})}{ac},$$

$$s_3(a,c)_\Lambda + s_3(c,a)_\Lambda = \frac{2s_2(a,c)_\Lambda + 2s_2(c,a)_\Lambda}{ac} - \frac{\alpha_1(\Lambda)(a^{q-1} + c^{q-1})}{a^3 c^3}.$$

In particular, if $\Lambda = K$, then $e_K(z) = z - z^q$, so that

$$s_1(a,c)_K + s_1(c,a)_K = -\frac{a^{q-1} + c^{q-1}}{ac},$$

$$s_3(a,c)_K + s_3(c,a)_K = \frac{2s_2(a,c)_K + 2s_2(c,a)_K}{ac} + \frac{a^{q-1} + c^{q-1}}{a^3 c^3}.$$

Acknowledgement. This work was partially supported by Grant-in-Aid for Scientific Research (No. 18540050), Japan Society for the Promotion of Science.

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