Duality of linking pairing in Arnold's singularities

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Abstract: As a new aspect of the Arnold strange duality among 14 unimodal singularities, we point out that there exists a duality in linking pairing on Seifert manifolds associated with singularities.

Key words: Linking pairing; Seifert manifold; singularity; Arnold's strange duality.

1. Introduction. We study the weighted homogeneous polynomials $f(x, y, z)$ in $\mathbf{C}^3$;

$$f\big(t^{d_1} x, t^{d_2} y, t^{d_3} z\big) = t^d f(x, y, z),$$

where (d_1, d_2, d_3) is the weight, and d is the degree. We suppose that the variety

$$V = \{f(x, y, z) = 0\}$$

has an isolated singularity at the origin. Classification of singularities has been widely studied [1] (see also *e.g.* [3]), and among them we pay attention to singularities listed in Tables I and II. These singularities in Tables I and II are respectively called the ADE singularities and the exceptional unimodal singularities.

Arnold observed that there exists a duality among 14 unimodal singularities [1]. This "strange duality" is related to the mirror symmetry from string theory, and it is interpreted from the viewpoint of the weight system [13,17]. Purpose of this Letter is to show that the strange duality can also be seen as a duality of linking pairing, which is a symmetric bilinear pairing on the torsion subgroup of the first homology group, on 3-manifolds associated with Arnold's singularities;

Theorem 1 (Duality of linking pairing)**.** *Let X and X^* be dual unimodal singularities in Arnold's sense, and let M and M^* be the Seifert manifolds associated with X and X^* respectively as* (2)*. Then M and M^* have the isomorphic first homology group H_1, and we have a duality of the linking pairing,*

$$\lambda_M = -\lambda_{M^*}. \tag{1}$$

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In the rest of this Letter, we briefly review Arnold's strange duality, and we give a proof of Theorem 1.

2. Arnold's strange duality. It is well known that isolated singularities in V are related to tessellations. Let Δ be the triangle with angles π/p_1, π/p_2, and π/p_3. The ADE singularities and the unimodal singularities in Tables I and II respectively correspond to quotient singularity associated with spherical and hyperbolic triangles. In the case of the hyperbolic triangle Δ, the triangle group Γ generated by the reflections with respect to the sides of Δ includes an invariant subgroup Γ' of finite index which acts on $\mathbf{H}$ without fixed points, and the quotient $\mathbf{H}/\Gamma'$ is a compact Riemann surface Σ. Then there exists a canonical morphism $\Sigma \to \mathbf{P}^1 = \mathbf{H}/\Gamma$. The singularity in Table II is isomorphic to this triangular singularity [4]. The triple (p_1, p_2, p_3) is called the Dolgachev number. It is known that the Milnor lattice $(H_2(F; \mathbf{Z}), \langle\rangle)$ of the unimodal singularities, where F is the Milnor fiber, is $T_{b_1,b_2,b_3} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Here T_{b_1,b_2,b_3} is the Coxeter–Dynkin diagram of star type with length b_1, b_2, b_3. The triple (b_1, b_2, b_3) is the Gabrielov number of the singularity [9].

Theorem (Arnold's strange duality [1])**.** *Let X be a singularity among Arnold's 14 unimodal singularities in Table II. Then there exists a unimodal singularity X^* whose Gabrielov number (b_1^*, b_2^*, b_3^*) coincides with the Dolgachev number (p_1, p_2, p_3) of X.*

One of interpretations of the strange duality follows from the weight systems associated with the unimodal singularities [13]. The weight system $W = (d_1, d_2, d_3; d)$ denotes a set of the weight (d_1, d_2, d_3)

Table I. ADE singularities

X	$f(x,y,z)$	(d_1,d_2,d_3)	d	(p_1,p_2,p_3)	$H_1(M;\mathbf{Z})$	λ_M	$\mathrm{CS}(M)$
D_{4K}	$x^{4K-1}+x\,y^2+z^2$	$(2,4\,K-2,4\,K-1)$	$8\,K-2$	$(2,2,4K-2)$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$	$\left\{-\dfrac{(2m+1)^2}{16K-8}\right\}$ for $0\le m\le 2K-2$
D_{4K+2}	$x^{4K+1}+x\,y^2+z^2$	$(2,4\,K,4\,K+1)$	$8\,K+2$	$(2,2,4K)$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\left(\frac{1}{2}\right)\oplus\left(\frac{1}{2}\right)$	$\left\{-\dfrac{(2m+1)^2}{16K}\right\}$ for $0\le m\le 2K-1$
D_{4K+1}	$x^{4K}+x\,y^2+z^2$	$(2,4\,K-1,4\,K)$	$8\,K$	$(2,2,4K-1)$	$\mathbf{Z}_4$	$\left(\frac{3}{4}\right)$	$\left\{-\dfrac{(2m+1)^2}{16K-4}\right\}$ for $0\le m\le 2K-2$
D_{4K+3}	$x^{4K+2}+x\,y^2+z^2$	$(2,4\,K+1,4\,K+2)$	$8\,K+4$	$(2,2,4K+1)$	$\mathbf{Z}_4$	$\left(\frac{1}{4}\right)$	$\left\{-\dfrac{(2m+1)^2}{16K+4}\right\}$ for $0\le m\le 2K-1$
E_6	$x^4+y^3+z^2$	$(3,4,6)$	12	$(2,3,3)$	$\mathbf{Z}_3$	$\left(\frac{2}{3}\right)$	$\left\{-\dfrac{1}{24}\right\}$
E_7	$x^3\,y+y^3+z^2$	$(4,6,9)$	18	$(2,3,4)$	$\mathbf{Z}_2$	$\left(\frac{1}{2}\right)$	$\left\{-\dfrac{1}{48},-\dfrac{25}{48}\right\}$
E_8	$x^5+y^3+z^2$	$(6,10,15)$	30	$(2,3,5)$	0	$\varnothing$	$\left\{-\dfrac{1}{120},-\dfrac{49}{120}\right\}$

Table II. Unimodal singularities

X	$f(x,y,z)$	(d_1,d_2,d_3)	d	Dolgachev (p_1,p_2,p_3)	Gabrielov (b_1,b_2,b_3)	$H_1(M;\mathbf{Z})$	λ_M	$\mathrm{CS}(M)$
E_{12}	$x^7+y^3+z^2$	$(6,14,21)$	42	$(2,3,7)$	$(2,3,7)$	0	$\varnothing$	$\left\{\frac{25}{168},-\frac{47}{168}\right\}$
E_{13}	$x^5y+y^3+z^2$	$(4,10,15)$	30	$(2,4,5)$	$(2,3,8)$	$\mathbf{Z}_2$	$\left(\frac{1}{2}\right)$	$\left\{\frac{9}{80},\frac{49}{80}\right\}$
E_{14}	$x^8+y^3+z^2$	$(3,8,12)$	24	$(3,3,4)$	$(2,3,9)$	$\mathbf{Z}_3$	$\left(\frac{2}{3}\right)$	$\left\{\frac{1}{12},\frac{25}{48}\right\}$
Z_{11}	$x^5+xy^3+z^2$	$(6,8,15)$	30	$(2,3,8)$	$(2,4,5)$	$\mathbf{Z}_2$	$\left(\frac{1}{2}\right)$	$\left\{\frac{25}{96},\frac{73}{96}\right\}$
Z_{12}	$x^4y+xy^3+z^2$	$(4,6,11)$	22	$(2,4,6)$	$(2,4,6)$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\left(\frac{1}{2}\right)\oplus\left(\frac{1}{2}\right)$	$\left\{\frac{3}{16},\frac{11}{16}\right\}$
Z_{13}	$x^6+xy^3+z^2$	$(3,5,9)$	18	$(3,3,5)$	$(2,4,7)$	$\mathbf{Z}_2\oplus\mathbf{Z}_3$	$\left(\frac{1}{2}\right)\oplus\left(\frac{2}{3}\right)$	$\left\{\frac{2}{15},\frac{19}{30}\right\}$
W_{12}	$x^5+y^4+z^2$	$(4,5,10)$	20	$(2,5,5)$	$(2,5,5)$	$\mathbf{Z}_5$	$\left(\frac{3}{5}\right)$	$\left\{\frac{5}{8},\frac{9}{40}\right\}$
W_{13}	$x^4y+y^4+z^2$	$(3,4,8)$	16	$(3,4,4)$	$(2,5,6)$	$\mathbf{Z}_8$	$\left(\frac{5}{8}\right)$	$\left\{\frac{1}{6},\frac{13}{24}\right\}$
Q_{10}	$x^4+y^3+xz^2$	$(6,8,9)$	24	$(2,3,9)$	$(3,3,4)$	$\mathbf{Z}_3$	$\left(\frac{1}{3}\right)$	$\left\{\frac{49}{72},\frac{25}{72},-\frac{5}{24}\right\}$
Q_{11}	$x^3y+y^3+xz^2$	$(4,6,7)$	18	$(2,4,7)$	$(3,3,5)$	$\mathbf{Z}_2\oplus\mathbf{Z}_3$	$\left(\frac{1}{2}\right)\oplus\left(\frac{1}{3}\right)$	$\left\{\frac{19}{112},\frac{75}{112},-\frac{29}{112},\frac{27}{112}\right\}$
Q_{12}	$x^5+y^3+xz^2$	$(3,5,6)$	15	$(3,3,6)$	$(3,3,6)$	$\mathbf{Z}_3\oplus\mathbf{Z}_3$	$\left(\frac{1}{3}\right)\oplus\left(\frac{2}{3}\right)$	$\left\{\frac{5}{24},\frac{17}{24},\frac{1}{6},\frac{2}{3}\right\}$
S_{11}	$x^4+y^2z+xz^2$	$(4,5,6)$	16	$(2,5,6)$	$(3,4,4)$	$\mathbf{Z}_8$	$\left(\frac{3}{8}\right)$	$\left\{\frac{7}{10},\frac{19}{30},\frac{3}{10}\right\}$
S_{12}	$x^3y+y^2z+xz^2$	$(3,4,5)$	13	$(3,4,5)$	$(3,4,5)$	$\mathbf{Z}_{13}$	$\left(\frac{5}{13}\right)$	$\left\{\frac{13}{60},\frac{37}{60},\frac{133}{240},\frac{157}{240}\right\}$
U_{12}	$x^4+y^3+z^3$	$(3,4,4)$	12	$(4,4,4)$	$(4,4,4)$	$\mathbf{Z}_4\oplus\mathbf{Z}_4$	$\begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{2}\end{pmatrix}$	$\left\{\frac{9}{16},\frac{1}{4}\right\}$

and the degree d of polynomials. For two weight systems, $W = (d_1, d_2, d_3; d)$ and $W^* = (d_1^*, d_2^*, d_3^*; d^*)$, a 3×3 matrix $\mathbf{Q}$ with non-negative elements is defined by

$$(d_1, d_2, d_3)\,\mathbf{Q} = (d, d, d), \qquad \mathbf{Q}\begin{pmatrix} d_1^* \\ d_2^* \\ d_3^* \end{pmatrix} = \begin{pmatrix} d^* \\ d^* \\ d^* \end{pmatrix}.$$

This matrix is called a weight magic square. Two weight systems, W and W^*, are dual if there exists a weight magic square $\mathbf{Q}$ for (W, W^*) with $|\det \mathbf{Q}| = d = d^*$.

Theorem (Kobayashi's duality [13])**.** *Let W be a weight system for Arnold's unimodal singularity X. Then there exists a dual weight system W^* for unimodal singularity X^* which is dual to X in Arnold's sense.*

Another aspect of the strange duality is due to [17]. A characteristic homeomorphism of the Milnor fiber F induces an automorphism, $C : H_2(F; \mathbf{Z}) \to H_2(F; \mathbf{Z})$, called the Milnor monodromy operator, and its eigenvalues are roots of unity. Thus the characteristic polynomial, $\phi(t) = \det(t - C)$, is written as

$$\phi(t) = \frac{1 - t^d}{1 - t} \prod_{i=1}^{3} \frac{1 - t^{p_i}}{1 - t^{d_i}} = \prod_{m|h} (t^m - 1)^{\chi_m},$$

where h is the order of C, and χ_m is the cyclotomic exponent. A dual polynomial $\phi^*(t)$ is defined by

$$\phi^*(t) = \prod_{k|h} \left(t^k - 1\right)^{-\chi_{h/k}}.$$

Theorem (Saito's duality [17])**.** *If $\phi(t)$ is the characteristic polynomial for Arnold's singularity X, a dual polynomial $\phi^*(t)$ is the characteristic polynomial for X^* which is dual to X in Arnold's sense.*

3. Duality of linking pairing. Let V be the variety $V = \{f(x, y, z) = 0\}$ for a weighted homogeneous polynomial $f(x, y, z)$ of weight (d_1, d_2, d_3). We have a natural $\mathbf{C}^*$-action on V given by

$$t(x, y, z) = \left(t^{d_1} x,\ t^{d_2} y,\ t^{d_3} z\right)$$

for $t \in \mathbf{C}^*$. To the isolated singularity of V at the origin, we associate the closed oriented 3-manifold M given by

$$M = V \cap S^5 \tag{2}$$

with a sufficiently small 5-sphere around the origin. With respect to the S^1-action induced from the $\mathbf{C}^*$-action on V, M is a Seifert manifold. As was studied in [15], its Seifert invariant is given as follows: The Seifert manifold M has 3 singular fibers along $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$, and the Seifert invariant $(g; b; (p_1, q_1), (p_2, q_2), (p_3, q_3))$ is calculated from a given f in the way written in [15]. In particular, for the f's in Tables I and II, we can show by calculating each case concretely that the Seifert invariant is $(0; -1; (p_1, 1), (p_2, 1), (p_3, 1))$ for the p_i's written in the tables.

We recall the definition of the linking pairing λ_M on a closed oriented 3-manifold M. The linking pairing is a symmetric bilinear form

$$\lambda_M : \operatorname{Tor} H_1(M; \mathbf{Z}) \otimes \operatorname{Tor} H_1(M; \mathbf{Z}) \to \mathbf{Q}/\mathbf{Z},$$

where $\operatorname{Tor} H_1(M; \mathbf{Z})$ denotes the torsion part of $H_1(M; \mathbf{Z})$. For $\boldsymbol{a}, \boldsymbol{a}' \in \operatorname{Tor} H_1(M; \mathbf{Z})$, we define $\lambda_M(\boldsymbol{a}, \boldsymbol{a}') \in \mathbf{Q}/\mathbf{Z}$ as follows: We choose a non-zero integer s such that $s\,\boldsymbol{a} = 0 \in H_1(M; \mathbf{Z})$, and set a 2-chain $\boldsymbol{B}$ which is bounded as $\partial \boldsymbol{B} = s\,\boldsymbol{a}$. We put

$$\lambda_M(\boldsymbol{a}, \boldsymbol{a}') = \frac{\#(\boldsymbol{B} \cdot \boldsymbol{a}')}{s} \mod \mathbf{Z}.$$

This is well defined, independently of the choices of s and $\boldsymbol{B}$. See [12] for the classification of linking pairings on 3-manifolds.

Proof of Theorem 1. We compute the linking pairing of the Seifert manifold M of type $(0; b; (p_1, q_1), (p_2, q_2), (p_3, q_3))$ as follows: We have

$$\begin{aligned} &H_1(M; \mathbf{Z}) \\ &\quad \cong \operatorname{span}_{\mathbf{Z}}\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{h}\} \Big/ \left\{ \begin{matrix} p_i\,\boldsymbol{x}_i + q_i\,\boldsymbol{h} \\ \text{for } i = 1, 2, 3 \\ \boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{x}_3 - b\,\boldsymbol{h} \end{matrix} \right\} \\ &\quad \cong \mathbf{Z}^4 / \mathbf{A}\,\mathbf{Z}^4, \end{aligned} \tag{3}$$

where

$$\mathbf{A} = \begin{pmatrix} p_1 & & & 1 \\ & p_2 & & 1 \\ & & p_3 & 1 \\ q_1 & q_2 & q_3 & -b \end{pmatrix}. \tag{4}$$

The order of $H_1(M; \mathbf{Z})$ is equal to $|\det \mathbf{A}|$. We suppose that $|\det \mathbf{A}| \neq 0$; this holds for all cases which we need in the following of this proof. Then we identify $\operatorname{Tor} H_1(M; \mathbf{Z})$ with $\mathbf{Z}^4 / \mathbf{A}\,\mathbf{Z}^4$. For $\boldsymbol{\ell}, \boldsymbol{\ell}' \in \mathbf{Z}^4$, we compute $\lambda_M(\boldsymbol{\ell}, \boldsymbol{\ell}')$ as follows: Here $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3, \ell_4)^T$ means a generator, $\ell_1\,\boldsymbol{x}_1 + \ell_2\,\boldsymbol{x}_2 + \ell_3\,\boldsymbol{x}_3 + \ell_4\,\boldsymbol{h}$, as (3). We thus have $s \in \mathbf{Z}_{\neq 0}$ and $\boldsymbol{m} =$

$(m_1, m_2, m_3, m_4)^T \in \mathbf{Z}^4$ such that $s\,\boldsymbol{\ell} = \mathbf{A}\,\boldsymbol{m}$. As m_i corresponds to the number of meridian disks of the i-th solid torus, we obtain by definition

$$\lambda_M(\boldsymbol{\ell}, \boldsymbol{\ell}') = \frac{1}{s}\,\boldsymbol{m}^T \begin{pmatrix} q_1 & & & \\ & q_2 & & \\ & & q_3 & \\ & & & 1 \end{pmatrix} \boldsymbol{\ell}'$$

$$= \boldsymbol{\ell}^T\,(\mathbf{A}')^{-1}\,\boldsymbol{\ell}', \tag{5}$$

where

$$\mathbf{A}' = \begin{pmatrix} \frac{p_1}{q_1} & & & 1 \\ & \frac{p_2}{q_2} & & 1 \\ & & \frac{p_3}{q_3} & 1 \\ 1 & 1 & 1 & -b \end{pmatrix}. \tag{6}$$

By use of an integral unimodular matrix $\mathbf{P}$, the matrix $(\mathbf{A}')^{-1}$ is block-diagonalized as $\mathbf{P}^T\,(\mathbf{A}')^{-1}\,\mathbf{P} = \boldsymbol{\Lambda} \oplus (\pm 1) \oplus \cdots \oplus (\pm 1) \mod \mathbf{Z}$.

We calculate λ_M for the cases in Table II. We take an example E_{14}. As we have $H_1(M;\mathbf{Z}) \cong \mathbf{Z}_3$ due to $|\det \mathbf{A}| = 3$, the linking pairing may be $(\frac{1}{3})$ or $(\frac{2}{3})$. Since we have $\boldsymbol{\ell}^T\,(\mathbf{A}')^{-1}\,\boldsymbol{\ell} = \frac{2}{3} \mod \mathbf{Z}$ with $\boldsymbol{\ell} = (1,0,0,0)^T$, we conclude that $\lambda_M = (\frac{2}{3})$. In the case of Z_{12}, we have from (3)

$$H_1(M;\mathbf{Z}) \cong \mathbf{Z}^4/\mathrm{span}_{\mathbf{Z}} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$
$$\cong \mathbf{Z}_2 \oplus \mathbf{Z}_2.$$

Taking generators, $\boldsymbol{\ell}_1 = (0,1,1,0)^T$ and $\boldsymbol{\ell}_2 = (0,0,1,0)^T$, we get $(\boldsymbol{\ell}_i{}^T\,(\mathbf{A}')^{-1}\,\boldsymbol{\ell}_j)_{i,j=1,2} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ mod $\mathbf{Z}$, which proves that the linking pairing is $(\frac{1}{2}) \oplus (\frac{1}{2})$. In this way we obtain symmetric matrices for M associated with the unimodal singularities as in Table II.

We show (1) for the cases in Table II. It is straightforward to verify the duality $E_{13} \leftrightarrow Z_{11}$, $E_{14} \leftrightarrow Q_{10}$, $Z_{13} \leftrightarrow Q_{11}$, $W_{13} \leftrightarrow S_{11}$, and the self-duality of E_{12}, Z_{12}, Q_{12}, because the linking pairing $(\frac{a}{b})$ is dual to $(\frac{-a}{b})$ in the sense of (1). In the cases of W_{12} and S_{12}, we note that $\frac{-3}{5}\,\ell^2 = \frac{3}{5}\,(2\,\ell)^2 \mod \mathbf{Z}$, and that $\frac{-5}{13}\,\ell^2 = \frac{5}{13}\,(5\,\ell)^2 \mod \mathbf{Z}$. For U_{12}, we use a unimodular matrix to find $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \mod \mathbf{Z}$.

This completes the proof. □

4. Concluding remarks. We have shown that there exists a duality of linking pairing on the Seifert manifolds associated with Arnold's 14 unimodal singularities. It should be noted that the ADE singularities in Table I do not have such duality of linking pairing while they are self-dual in both Kobayashi's and Saito's senses.

The Arnold strange duality has received renewed interests related to the mirror symmetry from string theory. Due to that the duality of weight system is related to the polar duality [2,7], it is discussed [5,6,8,16] that the Arnold strange duality is regarded as a two-dimensional analogue of the mirror symmetry of Calabi–Yau manifolds.

We observe a duality of linking pairing on the Seifert manifold M associated with Arnold's singularities X based on explicit computations of the SU(2) Witten–Reshetikhin–Turaev (WRT) invariant $\tau_N(M)$ for M. This result supports our decomposition conjecture proposed in [11]. It is noted that the Chern–Simons invariant CS(M) in Tables I and II is taken from asymptotic behaviors of the WRT invariants $\tau_N(M)$ in $N \to \infty$. See [10] for another aspect of the strange duality from the viewpoint of quantum invariants. See also [14] where invariant of 3-manifolds was constructed as a generalization of the WRT invariant at $N = 3$ based on linking matrices.

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