Discreteness of subgroups of $PU(1, n; \mathbf{C})$

By BaoHua XIE, YuePing JIANG, and Hua WANG

College of Mathematics and Econometrics, Hunan University, Changsha, 410082 Hunan, China

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Abstract: In this paper, we discuss three discreteness criterions of n-dimensional subgroup G of $PU(1, n; \mathbf{C})$. This generalize some discreteness criterions established by J. Gilman [3], S. Yang and A. Fang [9].

Key words: Discrete groups; dense group; regular elliptic elements.

1. Introduction. In [6], Jørgensen proved that a non-elementary subgroup G of $SL(2, \mathbf{C})$ is discrete if and only if all it's two-generator subgroups are discrete. W. Abikoff and A. Hass [1] generalized Jørgensen's result to isometries of higher-dimensional Möbius transformations. Furthermore, J. Gilman [3] and N. A. Isachenko [5] showed that the discreteness of all two-generator subgroups, where each generator is loxodromic, is enough to secure the discreteness of the group. Recently, on complex hyperbolic space, S. Yang and A. Fang showed that:

Theorem[9]. *Let G be a 2-dimensional subgroup in $PU(1, 2; \mathbf{C})$ containing elliptic elements, then G is discrete if and only if for each pair of elliptic elements f and g in G, the subgroup $\langle f, g \rangle$ is discrete.*

In this paper, we obtain three discreteness criterions by mainly concentrating on a n-dimensional subgroup G of $PU(1, n; \mathbf{C})$. Here G is a n-dimensional subgroup if G does not leave a point in $\partial \mathbf{H}^n_{\mathbf{C}}$ or a proper totally geodesic sub-manifold of $\mathbf{H}^n_{\mathbf{C}}$ invariant. The basic idea in proving our results is to note that the sets of loxodromic elements or regular elliptic elements are both open facing the identity element. Our main results are as follows:

Theorem 1.1. *Let G be a n-dimensional subgroup in $PU(1, n; \mathbf{C})$ containing elliptic elements, then G is discrete if and only if for each pair of regular elliptic element f and loxodromic element g in G, the subgroup $\langle f, g \rangle$ is discrete.*

Theorem 1.2. *Let G be a n-dimensional subgroup in $PU(1, n; \mathbf{C})$ and contains elliptic elements, then G is discrete if and only if for each pair of loxodromic elements f and g in G, the subgroup $\langle f, g \rangle$ is discrete.*

Theorem 1.3. *Let G be a n-dimensional subgroup in $PU(1, n; \mathbf{C})$ containing elliptic elements, then G is discrete if and only if for each pair of regular elliptic elements f and g in G, the subgroup $\langle f, g \rangle$ is discrete.*

2. Proof of Theorems. First, we recall some terminologies which are used in the paper. More details are given in [4,7,8]. Let $\mathbf{C}^{n,1}$ be a copy of the vector space $\mathbf{C}^{n+1}$ equipped with the Hermitian form $\langle \cdot, \cdot \rangle$:

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1\overline{v_1} + u_2\overline{v_2} + \cdots u_n\overline{v_n} - u_{n+1}\overline{v_{n+1}},$$

where $\mathbf{u}, \mathbf{v} \in \mathbf{C}^{n+1}$.

$\mathbf{H}^n_{\mathbf{C}}$ and its ideal boundary are respectively the projective images, in the complex projective plane $\mathbf{CP}^n$, of

$$V_- = \{\mathbf{v} \in \mathbf{C}^{n,1} | \langle \mathbf{v}, \mathbf{v} \rangle < 0\},$$
$$V_0 = \{\mathbf{v} \in \mathbf{C}^{n,1} | \langle \mathbf{v}, \mathbf{v} \rangle = 0\}.$$

The projection

$$(v_1, v_2, \cdots v_{n+1}) \to (v_1/v_{n+1}, \cdots v_n/v_{n+1})$$

takes V_-, V_0 respectively to the open unit ball and unit sphere in $\mathbf{C}^n$. Henceforth, we identify $\mathbf{H}^n_{\mathbf{C}}$ with an open unit ball. There exists a negative sectional curvature metric (Bergman metric) on $\mathbf{H}^n_{\mathbf{C}}$ which is invariant under complex projective automorphisms.

The full isometry group acting on $\mathbf{H}^n_{\mathbf{C}}$ is generated by holomorphic isometry group $PU(1, n; \mathbf{C})$ and complex conjugation. Moreover, each holomorphic isometry of $\mathbf{H}^n_{\mathbf{C}}$ is given by a matrix in $PU(1, n; \mathbf{C})$. Sometimes, it will be useful to consider $SU(1, n; \mathbf{C})$, the group of matrices with determinant 1 which are unitary with respect to $\langle \cdot, \cdot \rangle$. In fact, $SU(1, n; \mathbf{C})$ is a connected Lie

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group which $n+1$-fold covers $PU(1,n;\mathbf{C})$.

A non-trivial automorphism g of $\mathbf{H}^n_{\mathbf{C}}$ lifts to a unitary transformation $\widetilde{g}$ of $\mathbf{C}^{n,1}$ and the fixed points of g on $\mathbf{P}(\mathbf{C}^{n,1})$ correspond to eigenvectors of $\widetilde{g}$. By the Brouwer fixed points theorem, every automorphism of $\mathbf{H}^n_{\mathbf{C}}$ has a fixed point in $\mathbf{H}^n_{\mathbf{C}}\bigcup\partial\mathbf{H}^n_{\mathbf{C}}$. An automorphism g is called
(1) *elliptic* if it has a fixed point in $\mathbf{H}^n_{\mathbf{C}}$;
(2) *parabolic* if it has exactly one fixed point and the point lies on $\partial\mathbf{H}^n_{\mathbf{C}}$;
(3) *loxodromic* if it has exactly two fixed points and the points lie on $\partial\mathbf{H}^n_{\mathbf{C}}$.

In particular, if g is loxodromic, then $\widetilde{g}$ has one eigenvalue outside the unit circle and one eigenvalue inside the unit circle. The eigenvalues of the matrix corresponding to elliptic element all have norm 1. We say that an elliptic element g is regular elliptic if and only if its eigenvalues are distinct.

Theorem 6.2.1 in [4] tells us that the regular elliptic elements of $PU(1,n;\mathbf{C})$ form an open set. Next, we claim that the the loxodromic element of $PU(1,n;\mathbf{C})$ also form an open set. Let g be a loxodromic element of $PU(1,n;\mathbf{C})$ and g' is in a sufficient small neighborhood of g, the eigenvalues of $\widetilde{g'}$ approximating to the eigenvalues of $\widetilde{g}$ as the radius of neighborhood is sufficiently small. Then $\widetilde{g'}$ must have one eigenvalue outside the unit circle and one eigenvalue inside the unit circle. Therefore, g' must be a loxodromic element.

Denote $\mathbf{H}^n_{\mathbf{C}}\bigcup\partial\mathbf{H}^n_{\mathbf{C}}$ by $\overline{\mathbf{H}^n_{\mathbf{C}}}$, a subgroup G of $PU(1,n;\mathbf{C})$ is *elementary* if and only if G has a finite orbit in $\overline{\mathbf{H}^n_{\mathbf{C}}}$. We shall also divide the elementary subgroup of $PU(1,n;\mathbf{C})$ into three types. Let G be an elementary subgroup of $PU(1,n;\mathbf{C})$.

(1) G is said to be *elliptic type* if and only if G is a finite group consisting of elliptic elements and all its elements share a common fixed point in $\mathbf{H}^n_{\mathbf{C}}$;

(2) G is said to be *parabolic type* if and only if G fixes a point of $\partial\mathbf{H}^n_{\mathbf{C}}$ and has no other finite orbit in $\overline{\mathbf{H}^n_{\mathbf{C}}}$;

(3) G is said to be *loxodromic type* if and only if G has a cyclic subgroup of finite index generated by a loxodromic element with two fixed points a and b. If G contains elliptic elements, then it either fixes or exchanges a and b.

Lemma 2.1.[2]. *Let G be a closed subgroup of $U(1,n;F)$, where F is either $\mathbf{R}$ or $\mathbf{C}$, then one of the following is true*

(a) *G is discrete*;

(b) *The element of G have a common fixed point in $\overline{H^n(F)}$*;

(c) *G leaves invariant a proper, totally geodesic submanifolds*;

(d) *$F=\mathbf{C}$, and $G\supset SU(1,n;\mathbf{C})$*;

(e) *$G\supset U_0(1,n;F)$.*

Where $U_0(1,n;F)$ is the component of $U(1,n;F)$ containing identical element.

By the above Lemma 2.1, S. Chen and L. Greenberg obtained the following Proposition. For the convenience of the reader, we give a sketch here.

Proposition 2.2. *Let G be a n-dimension subgroup of $SU(1,n;\mathbf{C})$, then G is discrete or dense in $SU(1,n;\mathbf{C})$.*

Proof. Suppose that G is not discrete, then $\overline{G}$ is also not discrete. Since G is n-dimension subgroup of $SU(1,n;\mathbf{C})$, G can not satisfy (b), (c) of Lemma 2.1. As $G\subset\overline{G}$, $\overline{G}$ can also not satisfy (b), (c) of Lemma 2.1. We have $\overline{G}\supset SU(1,n;\mathbf{C})$ (i.e. (d) is true). $SU(1,n;\mathbf{C})$ is a closed Lie group, then $\overline{G}\subset SU(1,n;\mathbf{C})$. It follows that $\overline{G}=SU(1,n;\mathbf{C})$.

Lemma 2.3.[2]. *Let G be a n-dimensional subgroup in $PU(1,n;\mathbf{C})$, if the identity is not an accumulation point of the elliptic elements in G, then G is discrete.*

According to lemma 2.3, a direct result is that a n-dimensional subgroup in $PU(1,n;\mathbf{C})$ containing no elliptic element is discrete. Therefore, it is interested in discussing the discreteness of a n-dimensional subgroup in $PU(1,n;\mathbf{C})$ containing elliptic elements.

Proof of Theorem 1.1. We only need to show the "if" part. Suppose that G is not discrete, then G is dense in $PU(1,n,\mathbf{C})$ by Proposition 2.2. Since the set consisting of regular elliptic elements and the set consisting of loxodromic elements are open, we can choose a sequence $\{g_n\}$ consisting of regular elliptic elements and a sequence $\{f_n\}$ consisting of loxodromic elements such that $g_n\longrightarrow I$ and $f_n\longrightarrow I$ as $n\longrightarrow\infty$.

According to the assumption, $\langle f_m,g_k\rangle$ is discrete and then it is elementary by Margulis Lemma. Obviously, $\langle f_m,g_k\rangle$ can not be of parabolic type and elliptic type from the classification of elementary groups. If $\langle f_m,g_k\rangle$ is of loxodromic type, then g_k must swap the fixed points of f_m. This is impossible for $g_n\longrightarrow I$.

Proof of Theorem 1.2. Suppose that G is not discrete, G is dense in $PU(1,n,\mathbf{C})$ by Proposition 2.2. Since loxodromic elements of

$PU(1,n,\mathbf{C})$ form an open set in $PU(1,n,\mathbf{C})$ and G contains infinite loxodromic elements, there exists a sequence $\{g_n\} \subset G$ of distinct loxodromic elements such that $g_n \longrightarrow I(n \longrightarrow \infty)$. $\langle g_m, g_k \rangle$ is discrete from the assumption. Then it is elementary for sufficiently large m and k by Margulis Lemma. By the classification of elementary groups, $\langle g_m, g_k \rangle$ can not be elliptic type and parabolic type elementary group. If $\langle g_m, g_k \rangle$ is of loxodromic type, then g_m and g_k have the same fixed points. We can find an element f of the n-dimension subgroup such that f does not preserve the geodesic line determined by the fixed points of g_m or g_k. By the same argument as above, $\langle f g_m f^{-1}, g_k \rangle$ is of loxodromic type. This is a contradiction.

Proof of Theorem 1.3. Suppose that G is not discrete. Then G is dense in $PU(1,n,\mathbf{C})$ by Proposition 2.2. As the set consisting of regular elliptic elements form an open set, we can choose a sequence $\{g_n\}$ consisting of regular elliptic elements such that $g_n \longrightarrow I$ as $n \longrightarrow \infty$. The remainder of the proof is similar to the process of the case one in the proof of the main result of [9].

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