Principally tame regular sequences associated with the fourth Painlevé hierarchy with a large parameter

By Takashi Aoki*) and Naofumi Honda**)

(Communicated by Heisuke Hironaka, M.J.A., Feb. 12, 2008)

Abstract: The notion of principally tame regular sequences is introduced for systems of polynomials with a weight vector. As an application, construction of formal solutions is discussed for the systems of nonlinear differential equations which belong to the fourth Painlevé hierarchy with a large parameter.

Key words: Regular sequences; the exact WKB analysis; Painlevé hierarchies.

1. Introduction. The aim of this paper is to establish some criteria for systems of algebraic equations admitting finite number of solutions. As an application, we discuss the construction of formal solutions to differential equations which belong to the fourth Painlevé hierarchy with a large parameter.

The notion of tame regularity for systems of algebraic equations (or polynomials) was introduced in our previous paper [1], which guarantees finiteness of the number of solutions if the number of equations coincides with the dimension of the base space (cf. Definitions 2.1 and 2.2). In [1], the highest degree parts of the polynomials which define the systems play a role. For a sequence of polynomials, it is tame regular if the sequence which consists of the highest degree parts of the polynomials is tame regular [1; Theorem 8]. In this paper, we show that this is also true in the case where the degrees of polynomials are measured by using a weight vector. This generalization provides us much applicability: We prove the existence and the finiteness of the leading terms of formal solutions to a general member of the fourth Painlevé hierarchy with a large parameter. Once the leading terms are determined, the higher order terms of the formal solutions can be obtained successively under the condition that the Jacobi matrix of the system never vanishes. We give a sufficient condition for invertibility of the Jacobi matrix.

2000 Mathematics Subject Classification. Primary 34E05, 34E13, 34E15.

*) Kinki University, Department of Mathematics, 3-4-1, Kowakae, Higashi-Osaka, Osaka 577-8502, Japan.

**) Hokkaido University, Graduate School of Science, Kita 10, Nishi 8, Kita-ku, Sapporo, Hokkaido 060-0810, Japan.

2. Tame regular sequences and principally tame regular sequences. Let $\mathscr{O}$ denote the sheaf of rings of holomorphic functions on $\mathbf{C}^n$ and let $\hat{u}$ be a point in $\mathbf{C}^n$. For holomorphic functions $f_1, \dots, f_l$ on an open set $U \subset \mathbf{C}^n$, we denote the analytic set $\{u \in U; f_1(u) = \cdots = f_l(u) = 0\}$ by $V(f_1, \dots, f_l)$ and its germ at $\hat{u} \in U$ by $V(\hat{u}; f_1, \dots, f_l)$.

Definition 2.1 [1; Definition 2]**.** A sequence $\{f_1, f_2, \dots, f_l\}$ of elements in $\mathscr{O}_{\hat{u}}$ is said to be tame regular at $\hat{u}$ if for any integer k so that $0 \le k \le l-1$ and for any $(k+1)$ choice $f_{l_0}, f_{l_1}, \dots, f_{l_k}$ of elements in $\{f_1, f_2, \dots, f_l\}$, the element f_{l_k} is not a zero divisor on $\mathscr{O}_{\hat{u}}/(f_{l_0}, f_{l_1}, \dots, f_{l_{k-1}})$. Here $(f_{l_0}, f_{l_1}, \dots, f_{l_{k-1}})$ designates the ideal in $\mathscr{O}_{\hat{u}}$ generated by $f_{l_0}, f_{l_1}, \dots, f_{l_{k-1}}$.

If $\{f_1, f_2, \dots, f_l\}$ is tame regular at $\hat{u}$, then $\dim V(\hat{u}; f_{l_0}, f_{l_1}, \dots, f_{l_k}) = n - k - 1$ or $V(\hat{u}; f_{l_0}, f_{l_1}, \dots, f_{l_k}) = \emptyset$ for any $(k+1)$ choice $f_{l_0}, f_{l_1}, \dots, f_{l_k}$ of elements in $\{f_1, f_2, \dots, f_l\}$ [1; Theorem 2].

Let R denote the ring of polynomials of $u = (u_1, u_2, \dots, u_n)$ with complex coefficients.

Definition 2.2 [1; Definition 4]**.** A sequence $\{f_1, f_2, \dots, f_l\}$ of elements in R is said to be tame regular (in R) if for any k so that $0 \le k \le l-1$ and any $(k+1)$ choice $f_{l_0}, f_{l_1}, \dots, f_{l_k}$ of elements in $\{f_1, f_2, \dots, f_l\}$, the element f_{l_k} is not a zero divisor on $R/(f_{l_0}, f_{l_1}, \dots, f_{l_{k-1}})$. Here $(f_{l_0}, f_{l_1}, \dots, f_{l_{k-1}})$ denotes the ideal in R generated by $f_{l_0}, f_{l_1}, \dots, f_{l_{k-1}}$.

We note that $\{f_1, f_2, \dots, f_l\}$ is tame regular in R if and only if $\{f_1, f_2, \dots, f_l\}$ is tame regular at $\hat{u}$ for any $\hat{u} \in \mathbf{C}^n$ [1; Theorem 7].

Let $\boldsymbol{w} = (w_1, w_2, \dots, w_n)$ be an element of $\mathbf{N}^n$. For a monomial $cu^\alpha := cu_1^{\alpha_1} u_2^{\alpha_2} \cdots u_n^{\alpha_n}$ $(c \in \mathbf{C}^\times)$, we denote by $\deg_{\boldsymbol{w}}(cu^\alpha)$ the degree of the monomial cu^α

with respect to the weight $\boldsymbol{w}$. That is, we set

$$(1) \qquad \deg_{\boldsymbol{w}}(cu^{\alpha}) = \sum_{i=1}^{n} w_i \alpha_i.$$

For a polynomial $f(u) = \sum c_{\alpha} u^{\alpha}$ in R, we define the degree of f with respect to the weight $\boldsymbol{w}$ by

$$(2) \qquad \deg_{\boldsymbol{w}}(f) = \max_{\alpha} \deg_{\boldsymbol{w}}(c_{\alpha} u^{\alpha}).$$

Here we set, as usual, $\deg_{\boldsymbol{w}}(0) = -\infty$. If $\deg_{\boldsymbol{w}}(f) = m$, we set

$$(3) \qquad \sigma_{\boldsymbol{w}}(f) = \sum_{\deg_{\boldsymbol{w}}(c_{\alpha}u^{\alpha})=m} c_{\alpha} u^{\alpha}$$

and call it the principal part of f with respect to the weight $\boldsymbol{w}$. We set $\sigma = \sigma_{(1,1,\dots,1)}$. A polynomial $f \in R$ is said to be $\boldsymbol{w}$-homogeneous if $\sigma_{\boldsymbol{w}}(f) = f$.

Definition 2.3. A sequence $\{f_1, f_2, \dots, f_l\}$ of elements in R is said to be principally tame regular with respect to the weight $\boldsymbol{w}$ if the sequence $\{\sigma_{\boldsymbol{w}}(f_1), \sigma_{\boldsymbol{w}}(f_2), \dots, \sigma_{\boldsymbol{w}}(f_l)\}$ is tame regular.

Theorem 2.1. *Let $\{f_1, f_2, \dots, f_l\}$ be a sequence of polynomials in R. If there exists a weight vector $\boldsymbol{w} = (w_1, \dots, w_n) \in \mathbf{N}^n$ so that the sequence is principally tame regular with respect to $\boldsymbol{w}$, then the sequence is tame regular.*

Proof. For the special case where $w_j = 1$ for all $j = 1, 2, \dots, n$, this theorem is nothing but Theorem 8 in [1]. General case can be reduced to this special case by using the following

Lemma 2.2. *Let $f_1(u), \dots, f_l(u)$ $(l \le n)$ be elements in R and let*

$$(4) \qquad \Phi : \mathbf{C}^n_{\xi} \to \mathbf{C}^n_u$$

be an algebraic mapping defined by polynomials $\Phi_i(\xi)$ $(i = 1, \dots, n)$ of $\xi = (\xi_1, \dots, \xi_n)$. If Φ is a proper mapping, then the following two conditions are equivalent:

(1) *The sequence $\{f_1(u), \dots, f_l(u)\}$ is a tame regular sequence in $\mathbf{C}[u_1, \dots, u_n]$.*

(2) *The sequence $\{\Phi^* f_1(\xi), \dots, \Phi^* f_l(\xi)\}$ is a tame regular sequence in $\mathbf{C}[\xi_1, \dots, \xi_n]$.*

Here we set $\Phi^* f(\xi) = f(\Phi(\xi))$ for $f \in \mathbf{C}[u_1, \dots, u_n]$.

Proof. Since Φ is a proper map between two affine spaces of the same dimension, Φ is surjective and any fiber of it is finite. It is clear that $V(\Phi(\xi_0); f_{l_0}, \dots, f_{l_k}) = \emptyset$ is equivalent to $V(\xi_0; \Phi^* f_{l_0}, \dots, \Phi^* f_{l_k}) = \emptyset$ $(\xi_0 \in \mathbf{C}^n,\ k \le l)$. There exists a filtration

$$(5) \qquad \mathbf{C}^n_{\xi} = V_n \supset V_{n-1} \supset \cdots \supset V_0$$

consisting of locally closed sets in $\mathbf{C}^n_{\xi}$ so that the following conditions are satisfied for $i = 0, 1, \dots, n$:

(i) V_i is an analytic set of dimension at most i.

(ii) $V_i - V_{i-1}$ is a smooth manifold and if it is not empty, then $\dim(V_i - V_{i-1}) = i$.

(iii) $\Phi|_{V_i - V_{i-1}}$ is a smooth map.

Here we set $V_{-1} = \emptyset$. Using this filtration and taking an irreducible decomposition of $V(\xi_0; \Phi^* f_{l_0}, \dots, \Phi^* f_{l_k})$, we see that

$$(6) \qquad \begin{aligned} &\dim(V(\Phi(\xi_0); f_{l_0}, \dots, f_{l_k})) \\ &\quad = \dim(V(\xi_0; \Phi^* f_{l_0}, \dots, \Phi^* f_{l_k})) \end{aligned}$$

holds for any ξ_0 and for $k = 0, 1, \dots, l-1$. Hence it follows from Theorem 2 of [1] that for any $\xi_0 \in \mathbf{C}^n$, the following two conditions are equivalent:

(a) $\{f_1, \dots, f_l\}$ is tame regular at $\Phi(\xi_0)$.

(b) $\{\Phi^* f_1, \dots, \Phi^* f_l\}$ is tame regular at ξ_0.

Thus Lemma 1.2 follows from Theorem 7 in [1]. □

We go back to the proof of Theorem 2.1. For the weight vector $\boldsymbol{w} = (w_1, \dots, w_n)$, we take a proper mapping

$$(7) \qquad \Phi_{\boldsymbol{w}} : \mathbf{C}^n_{\xi} \to \mathbf{C}^n_u$$

by $\Phi_{\boldsymbol{w}}(\xi_1, \xi_2, \dots, \xi_n) = (\xi_1^{w_1}, \xi_2^{w_2}, \dots, \xi_n^{w_n})$. Since we have $\sigma(\Phi^*_{\boldsymbol{w}} f_i) = \Phi^*_{\boldsymbol{w}}(\sigma_{\boldsymbol{w}}(f_i))$, combining Lemma 2.2 and Theorem 8 in [1] yields Theorem 2.1. □

3. Tame regular sequences of weighted homogeneous polynomials with holomorphic coefficients. In this section, $\mathscr{O}$ denotes the sheaf of holomorphic functions on a complex manifold T. Let U be an open set in T. We consider the ring $\mathscr{O}(U)[u_1, \dots, u_n]$ of polynomials of $u_1, \dots, u_n$ with coefficients in $\mathscr{O}(U)$. Let $\boldsymbol{w} = (w_1, \dots, w_n) \in \mathbf{N}^n$ be a weight vector. For a polynomial of $u_1, \dots, u_n$ with coefficients in $\mathscr{O}(U)$, we can define its degree and principal part with respect to the weight $\boldsymbol{w}$ in the same way as for a polynomial with constant coefficients. A polynomial $f \in \mathscr{O}(U)[u_1, \dots, u_n]$ is said to be non-constant if $\deg_{\boldsymbol{w}}(f) > 0$. For given polynomials $f_1, \dots, f_l \in \mathscr{O}(U)[u_1, \dots, u_n]$ and for $t \in U$, we denote by $V_t(f_1, \dots, f_l)$ the algebraic set in $\mathbf{C}^n$

$$\{u \in \mathbf{C}^n_u;\ f_1(t, u) = \cdots = f_l(t, u) = 0\}$$

and by $V_t(\hat{u}; f_1, \dots, f_l)$ the germ of $V_t(f_1, \dots, f_l)$ at $\hat{u} \in \mathbf{C}^n_u$.

Theorem 3.1. *Let $\{f_1, \dots, f_l\}$ $(l \le n)$ be a sequence of non-constant $\boldsymbol{w}$-homogeneous elements in $\mathscr{O}(U)[u_1, \dots, u_n]$. Then the following two conditions are equivalent*:

(i) *For any* $t \in U$, $\{f_1, \ldots, f_l\}$ *is a tame regular sequence in* R.
(ii) *For any* $t \in U$, $\dim V_t(0; f_1, \ldots, f_l) = n - l$.

Remark 3.1. If $l = n$, the second condition is equivalent to $V_t(0; f_1, \ldots, f_l) = \{0\}$ for any $t \in U$.

Proof. Clearly the first statement implies the second. We prove the converse. For any $(k+1)$ choice $f_{l_0}, \ldots, f_{l_k}$ of elements in $\{f_1, \ldots, f_l\}$, the algebraic set

$$V_t(\Phi_{\boldsymbol{w}}^* f_{l_0}, \ldots, \Phi_{\boldsymbol{w}}^* f_{l_k})$$

is $\mathbf{C}^\times$-conic. Here $\Phi_{\boldsymbol{w}}$ is defined by (7). Hence we have

$$\begin{aligned}\dim V_t(\Phi_{\boldsymbol{w}}^* f_{l_0}, \ldots, \Phi_{\boldsymbol{w}}^* f_{l_k}) \\ = \dim V_t(0; \Phi_{\boldsymbol{w}}^* f_{l_0}, \ldots, \Phi_{\boldsymbol{w}}^* f_{l_k}) \\ = \dim V_t(0; f_{l_0}, \ldots, f_{l_k}) = n - k - 1.\end{aligned}$$

Combining this with

$$\dim V_t(\Phi_{\boldsymbol{w}}^* f_{l_0}, \ldots, \Phi_{\boldsymbol{w}}^* f_{l_k}) = \dim V_t(f_{l_0}, \ldots, f_{l_k})$$

and Theorems 2 and 7 in [1], we see that the first statement holds. □

Theorem 3.2. *Let* $\boldsymbol{w} \in \mathbf{N}^n$ *be a weight vector. Let* $\{f_1, \ldots, f_l\}$ $(l \le n)$ *be a sequence of non-constant elements in* $\mathscr{O}(U)[u_1, \ldots, u_n]$. *Let* $V = V(f_1, \ldots, f_l)$ *denote the analytic set in* $U \times \mathbf{C}^n$

$$\{(t, u) \in U \times \mathbf{C}^n; f_1(t, u) = \cdots = f_l(t, u) = 0\}$$

and $p : U \times \mathbf{C}^n \to U$ *the canonical projection with respect to* t. *If* $\{f_1, \ldots, f_l\}$ *is principally tame regular with respect to* $\boldsymbol{w}$ *for every* $t \in U$, *then we have the following*
(i) *For every fixed* $t \in U$, $\{f_1, \ldots, f_l\}$ *is a regular sequence.*
(ii) *If* $l = n$, *the restriction* $p|_V$ *of* p *to* V *is a proper map and each fiber* $(p|_V)^{-1}(t)$ *is a finite set.*

Proof. Taking the map $\Phi_{\boldsymbol{w}}$, we can assume $\boldsymbol{w} = (1, 1, \ldots, 1)$ from the beginning.
(i) By Theorem 2.1, $\{f_1, \ldots, f_l\}$ is a tame regular sequence. Hence it is sufficient to show that $V_t(f_1, \ldots, f_l)$ is not empty. Let us consider $V_t(f_1, \ldots, f_l)$ in $\mathbf{P}^n$ with homogeneous coordinates $(u, \eta) = (u_1, \ldots, u_n, \eta)$. We set

$$\tilde{f}_i(t, u, \eta) = \eta^{\deg f_i} f_i(t, u/\eta)$$

and

$$\tilde{V}_t(f_1, \ldots, f_l) = \{(u, \eta) \in \mathbf{P}^n; \tilde{f}_i(t, u, \eta) = 0 \text{ for all } i\}.$$

Then it is clear that $\tilde{V}_t(f_1, \ldots, f_l)$ is not empty. There exist homogeneous polynomials $g_i(t, u, \eta)$ so that for each i, we have

$$\tilde{f}_i(t, u, \eta) = \sigma(f_i)(t, u) + \eta g_i(t, u, \eta).$$

We set $H_\infty = \{(u, \eta); \eta = 0\}$ and $V_\infty = H_\infty \cap \tilde{V}_t(f_1, \ldots, f_l)$. Then we have

$$V_\infty = \{(u, 0); \sigma(f_i)(t, u) = 0 \text{ for all } i\}.$$

By the assumption, $\dim V_\infty = n - l - 1$ if $V_\infty \neq \emptyset$. Thus there exist $n - l$ hyperplanes $H_1, \ldots, H_{n-l}$ mutually transversal for which

$$V_\infty \cap H_1 \cap \cdots \cap H_{n-l} = \emptyset$$

holds. On the other hand,

$$\tilde{V}_t(f_1, \ldots, f_l) \cap H_1 \cap \cdots \cap H_{n-l} \neq \emptyset.$$

Thus we have $V_t(f_1, \ldots, f_l) \neq \emptyset$.
(ii) Suppose that $p|_V$ is not proper. Then there exist a point $\hat{t} \in T$ and a sequence $\{(t^{(k)}, u_1^{(k)}, \ldots, u_n^{(k)})\}$ $(k = 1, 2, \ldots)$ in V so that

$$t^{(k)} \to \hat{t} \quad \text{and} \quad \sum_{i=1}^n |u_i^{(k)}| \to \infty.$$

Thus there exists a point $(\hat{u}_1, \ldots, \hat{u}_n)$ satisfying

$$\frac{u_i^{(k)}}{\sum_{i=1}^n |u_i^{(k)}|} \to \hat{u}_i \quad \text{and} \quad \sum_{i=1}^n |\hat{u}_i| = 1.$$

Clearly $(\hat{t}, \hat{u}) = (\hat{t}, \hat{u}_1, \ldots, \hat{u}_n)$ entails

$$\sigma(f_i)(\hat{t}, \hat{u}) = 0$$

for $i = 1, \ldots, n$. This implies $\hat{u}_1 = \cdots = \hat{u}_n = 0$ and this is a contradiction. □

4. Tame regular sequences with parameters. Let U be an open set in $\mathbf{C}$ and A an open set in $\mathbf{C}^d$ with $d \ge n - 1$. We assume that U and A are connected. Let $\mathscr{O}(U \times A)$ denote the ring of holomorphic functions of $(t, \alpha) = (t, \alpha_1, \ldots, \alpha_d)$ defined in $U \times A$. For a sequence $\{f_1, \ldots, f_n\}$ of polynomials in $u_1, \ldots, u_n$ with coefficients in $\mathscr{O}(U \times A)$, we set

$$D(t, \alpha; u) = \det\left(\frac{\partial f_j}{\partial u_i}\right)_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}}. \tag{8}$$

Let I_{n-1} designate the set of subsets consisting of $n - 1$ elements of the set $\{1, 2, \ldots, d\}$ and for $I \in I_{n-1}$, we set

$$M_I(t, \alpha; u) = \det\left(\frac{\partial f_j}{\partial t}, \frac{\partial f_j}{\partial \alpha_i}\right)_{\substack{i \in I \\ j=1,\ldots,n}}. \tag{9}$$

For $t \in U$ and $\alpha \in A$, we denote respectively by $V_{t,\alpha}$ and by V_α the algebraic set

$$\{u \in \mathbf{C}^n; f_1(t,\alpha,u) = \cdots = f_n(t,\alpha,u) = 0\}$$

and the analytic set

$$\{(t,u) \in U \times \mathbf{C}^n; f_1(t,\alpha,u) = \cdots = f_n(t,\alpha,u) = 0\}.$$

For a fixed α, we set

$$T_\alpha = p(\{(t,u) \in V_\alpha; D(t,\alpha;u) = 0\}),$$

where $p : U \times \mathbf{C}^n \to U$ denotes the canonical projection.

Theorem 4.1. *Suppose that for each $(t,\alpha) \in U \times A$, the sequence $\{f_1,\dots,f_n\}$ is principally tame regular in $\mathbf{C}[u_1,\dots,u_n]$, and there exist a point $(\hat{t},\hat{\alpha}) \in U \times A$ and $I \in I_{n-1}$ for which $M_I(\hat{t},\hat{\alpha};u) \neq 0$ holds for each $u \in V_{\hat{t},\hat{\alpha}}$. Then T_α is a discrete set for each generic $\alpha \in A$. If $U = \mathbf{C}$ and every f_j is a polynomial in t, then T_α is a finite set.*

Proof. Since the proof goes in the same way as for any $d \geq n-1$, we will assume that $d = n-1$ and $I = \{1,2,\dots,n-1\}$. Let X denote the product space $U \times A \times \mathbf{C}^n$. We consider the analytic variety $\tilde{V}$ defined by $\{f_i\}$ in X:

$$\tilde{V} = \{(t,\alpha,u) \in X; f_1 = \cdots = f_n = 0\}.$$

Let $\tilde{p} : X \to U \times A$ denote the canonical projection. By the assumption, $\tilde{p}|_{\tilde{V}} : \tilde{V} \to U \times A$ is proper and each fiber of it is finite. We set

$$E = \tilde{p}(\tilde{V} \cap \{(t,\alpha,u) \in X; D(t,\alpha,u) = 0\}).$$

Since $\tilde{p}|_{\tilde{V}}$ is proper, E is an analytic set in $U \times A$.

Lemma 4.2. *Codimension of E in $U \times A$ is greater than or equal to* 1.

Proof. Let $\tilde{u}^{(1)},\dots,\tilde{u}^{(l)} \in \mathbf{C}^n$ be the solutions of $f_1(\hat{t},\hat{\alpha},u) = \cdots = f_n(\hat{t},\hat{\alpha},u) = 0$. By the assumption, there exist holomorphic functions $g_i^{(k)}(u)$ $(i = 1,\dots,n)$ in a neighborhood $W^{(k)} \subset \mathbf{C}_u^n$ of $\tilde{u}^{(k)}$ for $k = 1,\dots,l$ for which $\tilde{V}$ is expressed in the form $g_1^{(k)}(u) = t$ and $g_i^{(k)}(u) = \alpha_{i-1}$ $(i = 2,\dots,n)$ near $(\hat{t},\hat{\alpha},\tilde{u}^{(k)})$. Then we have

$$\text{(10)}\quad \det\left(\frac{\partial g_i^{(k)}}{\partial u_j}\right)\bigg|_{u=\tilde{u}^{(k)}} \times M_I(\hat{t},\hat{\alpha},\tilde{u}^{(k)}) = D(\hat{t},\hat{\alpha},\tilde{u}^{(k)}).$$

Let $\Psi^{(k)} : W^{(k)} \to \mathbf{C}_{t,\alpha}^n$ be a mapping defined by $\Psi^{(k)}(u) = (g_1^{(k)}(u),\dots,g_n^{(k)}(u))$. By the Sard theorem, the image of the set of all critical points of $\Psi^{(k)}$ has measure zero. Since $\Psi^{(k)}$ is proper, the image is an analytic set with codimension greater than or equal to 1. Thus there exist holomorphic functions $\omega^{(k)}(t,\alpha) \neq 0$ defined near $(\hat{t},\hat{\alpha})$ for which the image is contained in the set defined by $\omega^{(k)}(t,\alpha) = 0$. If we take a point (t,α) sufficiently close to $(\hat{t},\hat{\alpha})$ so that $\prod_{1 \leq k \leq l} \omega^{(k)}(t,\alpha) \neq 0$. Then $(t,\alpha) \notin E$. Since $U \times A$ is connected, the codimension of E is greater than or equal to 1. □

For $t \in U$, we set $E_t = \{\alpha \in A; (t,\alpha) \in E\}$ and $E_{\mathrm{ex}} = \bigcap_{t \in U} E_t$. Then, by Lemma 4.2, E_{ex} is an analytic set with codimension greater than or equal to 1. Hence for any $\tilde{\alpha} \notin E_{\mathrm{ex}}$, the set $E_{\mathrm{tp}} = \{t \in U; (t,\tilde{\alpha}) \in E\}$ is discrete. If f_i are polynomials in t, the number of irreducible components of $\tilde{V} \cap \{D = 0, \alpha = \tilde{\alpha}\}$ is finite. Hence E_{tp} is finite. □

5. Construction of formal solutions to the fourth Painlevé hierarchy with a large parameter. The fourth Painlevé hierarchy was introduced by [2] and it was investigated from the viewpoint of the exact WKB analysis by [4–6]. In the exact WKB analysis, a large parameter η is introduced and considering formal solutions that have expansion in the negative powers of η is a starting point of the analysis. In these papers, however, existence of such solutions is assumed. As an application of the results obtained in the previous sections, we prove that the assumption is correct in general.

We employ the formulation given in [5] with a slight modification. For $m = 1,2,\dots$, the m-th member $(P_{\mathrm{IV}})_m$ of the fourth Painlevé hierarchy has the following form:

$$\begin{cases} \eta^{-1}\partial_t X_m = 2Y_m + uX_m + g - 2\alpha, \\ \eta^{-1}X_m\partial_t Y_m = -vX_m^2 + (Y_m + \frac{g}{2} - \alpha)^2 - \frac{\beta^2}{4}, \end{cases}$$

where α, β, g are arbitrary constants and $X_m = K_m/2^m + gt$, $Y_m = L_m/2^m$ with polynomials K_m and L_m of unknown functions u, v and their derivatives $u' = \partial_t u$ and $v' = \partial_t v$ defined recursively by

$$\text{(11)}\quad \eta^{-1}\partial_t\begin{pmatrix} K_{j+1} \\ L_{j+1} \end{pmatrix} = P\begin{pmatrix} K_j \\ L_j \end{pmatrix}, \begin{pmatrix} K_0 \\ L_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Here we set

$$P = \begin{pmatrix} \eta^{-1}(u' + u\partial_t) - \eta^{-2}\partial_t^2 & \eta^{-1}2\partial_t \\ \eta^{-1}(2v\partial_t + v') & \eta^{-1}u\partial_t + \eta^{-2}\partial_t^2 \end{pmatrix}.$$

We look for formal solutions that have expansion in powers of η^{-1}. We put the expressions

$$\text{(12)}\qquad u = \sum_{k=0}^{\infty} \eta^{-k}u_k \text{ and } v = \sum_{k=0}^{\infty} \eta^{-k}v_k$$

into these equations and compare the coefficients of like powers of η^{-1}. Then we have the following system of algebraic equations for the leading terms u_0 and v_0:

$$(13)\qquad \begin{cases} 2Y_{m,0} + u_0 X_{m,0} + g - 2\alpha = 0, \\ -v_0 X_{m,0}^2 + \left(Y_{m,0} + \dfrac{g}{2} - \alpha\right)^2 - \dfrac{\beta^2}{4} = 0. \end{cases}$$

Here $X_{m,0}$ and $Y_{m,0}$ are polynomials of u_0, v_0 defined by $X_{m,0} = K_{m,0}/2^m + gt$, $Y_{m,0} = L_{m,0}/2^m$ and

$$(14)\qquad \begin{cases} \partial_t K_{j+1,0} = \partial_t(u_0 K_{j,0} + 2L_{j,0}), \\ \partial_t L_{j+1,0} = \partial_t(v_0 K_{j,0}) + v_0 \partial_t K_{j,0} + u_0 \partial_t L_{j,0} \end{cases}$$

with $(K_{0,0}, L_{0,0}) = (1,0)$. This recursive relation can be solved with ambiguity of integration constants, which are taken to be zero (see [5] for details). We set $f_1 = 2Y_{m,0} + u_0 X_{m,0} + g - 2\alpha$ and $f_2 = -v_0 X_{m,0}^2 + (Y_{m,0} + g/2 - \alpha)^2 - \beta^2/4$. These are polynomials of $u_0,\ v_0,\ t$.

Theorem 5.1. *For every $t \in \mathbf{C}$, $\{f_1, f_2\}$ is a principally tame regular sequence in $\mathbf{C}[u_0, v_0]$ with respect to the weight vector $(1,2)$. Hence (13) admits a finite number of solutions for every $t \in \mathbf{C}$.*

Proof. By the first equation of (14), we have

$$(15)\qquad K_{j+1,0} = u_0 K_{j,0} + 2L_{j,0}.$$

The second equation can be solved under the integrability condition $\partial K_{j,0}/\partial u_0 = \partial L_{j,0}/\partial v_0$:

$$(16)\qquad L_{j+1,0} = 2v_0 K_{j,0} + u_0 L_{j,0} - \int_0^{v_0} K_{j,0}\, dv.$$

Here $L_{j,0}(u_0, 0) = 0$ should be satisfied and the integrability condition for $K_{j+1,0}$ and $L_{j+1,0}$ requires

$$(17)\qquad v_0 \frac{\partial K_{j,0}}{\partial v_0} = \frac{\partial L_{j,0}}{\partial u_0}.$$

Note that the initial conditions $(K_{0,0}, L_{0,0})$ satisfy the integrability condition, $L_{0,0} = 0$ and (17). Thus we can solve $K_{j,0}$ and $L_{j,0}$ successively retaining the integrability condition. Combining (15) and (16), we have

$$(18)\qquad K_{j+1,0} = u_0 K_{j,0} + 2\frac{\partial}{\partial u_0}\int_0^{v_0} K_{j,0}\, dv.$$

This yields the relation

$$(19)\qquad \frac{\partial K_{j+1,0}}{\partial u_0} = (j+1) K_{j,0}.$$

If we write $K_{j,0} = \sum_{j,k} a_{jk}(v_0) u_0^k$, we have recurrent relations

$$(20)\qquad \begin{cases} a_{j+1,k+1} = \dfrac{j+1}{k+1} a_{j,k}, \\ a_{j+1,0} = 2\displaystyle\int_0^{v_0} a_{j,1}(v_0)\, dv_0 \end{cases}$$

with initial conditions $a_{0,0} = 1,\ a_{1,0} = 0$. Using these relations, we find the following explicit form for $K_{m,0}$:

$$(21)\qquad K_{m,0} = \sum_{0 \le l \le m/2} \frac{m!}{(m-2l)!\, l!^2} u_0^{m-2l} v_0^l$$

and hence

$$(22)\qquad L_{m,0} = \sum_{1 \le l \le (m+1)/2} \frac{m!\, u_0^{m+1-2l} v_0^l}{(m-2l+1)!\, l!\, (l-1)!}.$$

The sum of all $K_{m,0}$ has the following explicit form:

$$(23)\qquad \sum_{m=0}^{\infty} K_{m,0} = \frac{1}{\sqrt{1 - 2u_0 + (1-4z)u_0^2}},$$

where we set $z = v_0/u_0^2$. Comparing this with the generating function for the Legendre polynomials

$$\sum_{m=0}^{\infty} P_m(x) s^m = \frac{1}{\sqrt{1 - 2xs + s^2}},$$

we have the following expression of $K_{m,0}$ in terms of the m-th Legendre polynomial P_m:

$$(24)\qquad K_{m,0} = (u_0\sqrt{1-4z})^m P_m\left(\frac{1}{\sqrt{1-4z}}\right).$$

Lemma 5.2. *If $K_{m,0} = L_{m,0} = 0$, then $u_0 = v_0 = 0$.*

Proof. The assumptions imply $K_{m+1,0} = 0$ because we have (15). If $u_0 v_0 = 0$, (21) yields $u_0 = v_0 = 0$. Suppose that $u_0 v_0$ does not equal zero. $P_m(x)$ is an odd (resp. even) polynomial if m is odd (resp. even). It is well known that all the roots of $P_m(x) = 0$ are simple, contained in the interval $-1 < x < 1$ and the roots of $P_m(x) = 0$ separate the roots of $P_{m+1}(x) = 0$. In particular, $P_m(x)$ has $[m/2]$ roots in $0 < x < 1$. For a fixed $u_0 \neq 0$, $K_{m,0}$ is a polynomial of z of degree $[m/2]$ and all of its roots are contained in $z < 0$ which separate the roots of $K_{m+1,0} = 0$. Thus the system of algebraic equations $K_{m,0} = K_{m+1,0} = 0$ does not have a root if $u_0 v_0 \neq 0$. □

We continue the proof of Theorem 5.1. We assign the weight 1 for u_0 and 2 for v_0. That is, we take a weight vector $\boldsymbol{w} = (1,2)$. Since $2^m \sigma_{\boldsymbol{w}}(f_1) = 2L_{m,0} + u_0 K_{m,0}$ and $2^{2m}\sigma_{\boldsymbol{w}}(f_2) = -v_0 K_{m,0}^2 + L_{m,0}^2$, it is sufficient to show that

$$(25) \qquad \begin{cases} 2L_{m,0} + u_0 K_{m,0} = 0, \\ -v_0 K_{m,0}^2 + L_{m,0}^2 = 0 \end{cases}$$

implies $u_0 = v_0 = 0$. Eliminating $L_{m,0}$, we have

$$(26) \qquad \left(\frac{u_0^2}{4} - v_0\right) K_{m,0}^2 = 0.$$

If $v_0 = u_0^2/4$, we can eliminate v_0 in the first equation of (25) and we have $u_0 = 0$. If $K_{m,0} = 0$, we have $L_{m,0} = 0$ and hence $u_0 = v_0 = 0$ by Lemma 5.2. By Theorem 3.1, we obtain Theorem 5.1. □

It follows from the second statement of Theorem 3.2 that there exist finite numbers of leading terms u_0, v_0 which are algebraic functions of t. To construct higher order terms u_j, v_j $(j \geq 1)$ of (12), we have to see the Jacobi matrix

$$(27) \qquad D = \begin{pmatrix} \dfrac{\partial f_1}{\partial u_0} & \dfrac{\partial f_1}{\partial v_0} \\ \dfrac{\partial f_2}{\partial u_0} & \dfrac{\partial f_2}{\partial v_0} \end{pmatrix}$$

is invertible at any point in $V_t(f_1, f_2)$ for generic t. Taking I of the theorem so that M_I becomes the Jacobi matrix with respect to variables t and β, we apply Theorem 4.1. Since

$$\begin{aligned} M_I(t, \alpha, \beta, g; u_0, v_0) &= \det \begin{pmatrix} u_0 g & 0 \\ -2v_0 X_{m,0} g & -\frac{\beta}{2} \end{pmatrix} \\ &= -\frac{u_0 \beta g}{2}, \end{aligned}$$

we can take $\hat{\alpha}$, $\hat{\beta}$ and $\hat{g}$ so that $\hat{\beta}\hat{g} \neq 0$ and $\hat{g} - 2\hat{\alpha} \neq 0$ hold. Next we fix $t = 0$. Then we can see that $M_I(t, \hat{\alpha}, \hat{\beta}, \hat{g}; u_0, v_0)$ never vanishes for any solution $(u_0, v_0) \in V_{0,\hat{\alpha},\hat{\beta},\hat{g}}$. Thus D is invertible for generic t and we can determine u_j and v_j $(j \geq 1)$ successively once we fix the leading term (u_0, v_0).

Remark 5.1. Our discussion is based on the formulation of the fourth Painlevé hierarchy given in [2,3,5]. Another formulation is given in [4] (see also [6]). In general, the hierarchy includes integration constants which are chosen to be zero in our discussion. This specialization does not restrict applicability of our discussion for the general case which includes integration constants. The algebraic equations for the leading terms of the systems given in [4] are obtained if we replace (15) by

$$K_{j+1,0} = u_0 K_{j,0} + 2L_{j,0} + c_j$$

for some integration constant c_j. Our discussion can be applied, however, to prove the existence and the finiteness of the leading terms of formal solutions to nonlinear differential equations which belongs to the generalized fourth Painlevé hierarchies with a large parameter given in [4,6] because the terms containing the integration constants never have the highest weight degree and they do not affect the principal parts. That is, the principal parts of the leading terms (with respect to η^{-1}) of $\mathcal{K}_m$ and $\mathcal{L}_m$ of (3.2) in [4] are exactly the same as those of $K_{m,0}$ and $L_{m,0}$, respectively and the formulation given in [6] is equivalent to that in [4]. Moreover Theorem 4.1 also holds for the generalized systems. To see this, it is enough to verify the condition $M_I \neq 0$ for a parameter with the integration constants being zero. In fact, for such a parameter the algebraic equations with respect to the leading terms coincide with those in this paper. Hence we can construct formal solutions for the systems given in [4] and [6].

Acknowledgements. The authors thank Dr. T. Koike who kindly let them know the results in [4,6] and the relation between these papers and [6].

References

[1] T. Aoki and N. Honda, Regular sequence associated with the Noumi-Yamada equations with a large parameter, Algebraic Analysis of Differential Equations, Springer, 2008, pp. 45–53.

[2] P. R. Gordoa, N. Joshi and A. Pickering, On a generalized $2+1$ dispersive water wave hierarchy, Publ. Res. Inst. Math. Sci. **37** (2001), no. 3, 327–347.

[3] P. R. Gordoa, N. Joshi and A. Pickering, Second and fourth Painlevé hierarchies and Jimbo-Miwa linear problems, J. Math. Phys. **47** (2006), no. 7, 073504, 16 pp.

[4] T. Koike, On new expressions of the Painlevé hierarchies, Algebraic Analysis and the Exact WKB Analysis for Systems of Differential Equations, RIMS Kôkyûroku Bessatsu **B5** (2008), 137–182.

[5] T. Koike and Y. Nishikawa, On exact WKB analysis for the fourth Painlevé hierarchy, RIMS Kôkyûroku **1424** (2005), 160–169.

[6] T. Koike, Y. Nishikawa and Y. Takei, On the Stokes geometry of the fourth Painlevé hierarchy in exact WKB analysis. (in preparation).