A characterization of Möbius transformations

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Abstract: In this note we will show that an injection of the hyperbolic plane is Möbius if and only if for some $0 < \theta < \pi$, f preserves triangles with an interior angle equal to θ.

Key words: Möbius transformations; hyperbolic triangle; Lambert quatrilaterals.

1. Introduction. Möbius transformations have many beautiful properties. For example, a map is Möbius if and only if it preserves cross ratios. As for the geometrical aspect, circle-preserving is the most well-known characterization of Möbius transformations. Carathédory [2] proved that any injective mapping of a domain D of C to C is the restriction of a Möbius transformation if the image of any circle contained with its interior in D is itself a circle. This theorem is generalized by Höfer [7] to arbitrary dimensions, where circles are replaced by hyperspheres naturally. In these kinds of characterizations, there are no regularity assumptions such as differentiability or even continuity. This is the theme of a modern field of geometric research which is called characterizations of geometrical mappings under mild hypotheses; see [7] and the references therein. In [5], Haruki and Rassias gave a new characterization of Möbius transformations by using Apollonius quadrilaterals. Recall that the quadrilateral $ABCD$ is said to be an Apollonius quadrilateral if $\overline{AB}\cdot\overline{CD} = \overline{BC}\cdot\overline{DA}$ holds, where $\overline{AB}$ denotes the length of the line segment joining A and B. Then a quadrilateral is an Apollonius quadrilateral if the absolute cross ratio of its four vertices equals 1. Haruki and Rassias proved that if f is meromorphic and if f sends Apollonius quadrilaterals to Apollonius quadrilaterals, then f is Möbius. See [3,4,6] for other related results.

On the other hand, Möbius transformations are closely related to hyperbolic geometry since they act as isometries on hyperbolic space (see Appendix for a brief introduction of Möbius transformations and hyperbolic geometry). There are also many discussions in this direction. For example, Ungar [8] introduced the so-called Möbius addition in the open unit disk and even in the unit ball of any real inner product space by employing Möbius transformations. Recently, in [9], we considered the hyperbolic disk $B^2 = \{z|\ |z| < 1\}$ and presented a new characterization of Möbius transformations on B^2 by using a class of hyperbolic geometric objects as follows:

Theorem A. *Suppose $f : B^2 \to B^2$ is a continuous injection. Then f is Möbius if and only if f preserves Lambert quadrilaterals.*

According to [1], the Lambert quadrilateral is a hyperbolic quadrilateral which has exactly three right interior angles.

The purpose of this note is to show the continuity assumption in Theorem A can be removed. Furthermore, we will prove that hyperbolic triangles having angles with a fixed value can be used to characterize Möbius transformations acting on B^2, too. Specifically, we have

Theorem 1. *Suppose $f : B^2 \to B^2$ is an injection. Then f is Möbius if and only if for some $0 < \theta < \pi$, f preserves triangles with an interior angle equal to θ.*

2. Proof of Theorem 1. We denote by a prime the image under f, by $[A, B]$ the geodesic segment connecting A and B, by AB the geodesic through A and B, by $A_1A_2\cdots A_n$ the hyperbolic polygon with ordered vertices $A_1, A_2, \cdots, A_n$, and by $\angle BAC$ the angle between $[A, B]$ and $[A, C]$.

The following is useful for us, cf. [1, Theorem 7.16.2].

Proposition 2.1. *Let $\theta_1, \cdots, \theta_n$ be any ordered n-tuple with $0 \le \theta_j < (n-2)\pi, j = 1, \cdots, n$. Then there exists a polygon P with interior angles $\theta_1, \cdots, \theta_n$, occurring in this order around ∂P, if and only if $\theta_1 + \cdots + \theta_n < (n-2)\pi$.*

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Lemma 2.2. *Let $f : B^2 \to B^2$ be a continuous injection. If f preserves right triangles, then it is Möbius.*

Proof. Let ABC be a triangle with $\angle BAC = \dfrac{\pi}{2}$. Claim that $\angle B'A'C' = \dfrac{\pi}{2}$. Otherwise, we may assume that $\angle A'B'C' = \dfrac{\pi}{2}$. Then for any point $D \in [A, C]$, it is easy to see ABD is a right triangle, while the triangle $A'B'D'$ not. This is a contradiction.

Now given arbitrary Lambert quadrilateral $ABCD$, with $\angle BCD < \dfrac{\pi}{2}$. Obviously, there exist three right triangles DAB, ABC and ADC by the definition of Lambert quadrilaterals. The preceding arguments shows that $\angle B'A'D' = \angle A'B'C' = \angle A'D'C' = \dfrac{\pi}{2}$, i.e., $A'B'C'D'$ is also a Lambert quadrilateral. Hence f is Möbius by Theorem A. □

In the following lemmas, we always assume that $0 < \theta < \dfrac{\pi}{2}$. Then by Proposition 2.1, there exist quadrilaterals having interior angles $\dfrac{\pi}{2}, \theta, \theta, \theta$ and $\pi - \theta, \theta, \theta, \theta$, respectively, which we shall use later.

Lemma 2.3. *Let $f : B^2 \to B^2$ be a continuous injection and preserve quadrilaterals having interior angles equal to $\dfrac{\pi}{2}, \theta, \theta, \theta$. Then f is Möbius.*

Proof. By Lemma 2.2, we only need to show that f preserves right angles. Otherwise, for some quadrilateral $ABCD$, where $\angle A = \dfrac{\pi}{2}$ and $\angle B = \angle C = \angle D = \theta$, the angle at the vertex A' is not right, i.e., $\angle D'A'B' = \theta$.

Since both $\angle ABC$ and $\angle ACB$ are strictly less than $\dfrac{\pi}{2}$, there exists a point $E \in [B, C]$ such that the geodesic AE is orthogonal to BC. Thus $\angle BAE < \dfrac{\pi}{2} - \theta$ and then $\angle DAE > \theta$ because the angle sum of a hyperbolic triangle is less than π. By the same manner, for the point F on the geodesic ray DA such that $\angle FBC = \dfrac{\pi}{2}$, we have $\angle DFB < \theta$. Hence there exist $G \in [A, F]$ and $H \in [B, E]$, such that $\angle CHG = \dfrac{\pi}{2}$ and $\angle DGH = \theta$. By the assumption, $C'D'G'H'$ is also a quadrilateral with angles $\dfrac{\pi}{2}, \theta, \theta, \theta$. Then $\angle D'G'H' \geq \theta$. Note that $\angle D'A'B' = \theta$. However, it is easy to see $\angle D'A'B' > \angle D'G'H'$ since $\angle D'A'B' + \angle G'A'B' = \pi > \angle D'G'H' + \angle G'A'B'$. This is a contradiction. □

Lemma 2.4. *Let $f : B^2 \to B^2$ be a continuous injection. If f preserves triangles with an interior angle equal to θ, then it is Möbius.*

Proof. First we shall show that for a given triangle ABC with $\angle ABC = \theta$, $\angle A'B'C'$ also equals θ. Construct a triangle ABD such that $\angle ABD = \pi - \theta > \dfrac{\pi}{2}$. As in the proof of Lemma 2.2, it follows that $\angle A'B'D' = \pi - \theta$. Note that f preserves triangles implies it preserves geodesics. Then it follows that $\angle A'B'C' = \theta$.

Now we want to show f preserves quadrilaterals with angles $\dfrac{\pi}{2}, \theta, \theta, \theta$ and hence f is Möbius by using Lemma 2.3.

Suppose this is not the case for the quadrilateral $ABCD$, where $\angle A = \dfrac{\pi}{2}$ and $\angle B = \angle C = \angle D = \theta$. By Proposition 2.1, there exists a quadrilateral $BCEF$, such that $\angle CBF = \pi - \theta$ and $\angle F = \angle E = \angle BCE = \theta$. Reflecting in the geodesic AF, we get two quadrilaterals ABC_1D_1 and BC_1E_1F, satisfying $\angle BAD_1 = \dfrac{\pi}{2}$, $\angle C_1BF = \pi - \theta$ and all the left interior angles equal θ.

Consider the image under f of these four quadrilaterals. Note that f preserves angles equal to θ and $\pi - \theta$, respectively, since f preserves geodesics. Reflecting $B'C'E'F'$ in the geodesic $A'F'$, the quadrilateral we get is exactly $B'C_1'E_1'F'$ because they have the same interior angles and then we can use the properties of θ-transversals (cf. [1, §7.26]). Thus the image of $A'B'C'D'$ under the reflection in $A'F'$ is $A'B'C_1'G'$ for some point G'. Since $\angle B'C_1'G' = \angle B'C_1'D_1' = \theta$, we have $G' \in C_1'D_1'$. However, $\angle A'G'C_1' = \angle A'D_1'C_1' = \theta$. This is a contradiction. □

Remark. From the proof of Lemma 2.4, we know that a by-product of preserving angles equal to θ is that f also preserves right angle. This is in fact because f preserves geodesics, i.e., angles equal to π. Thus by the same manner, induction deduces that f preserves angles equal to $\dfrac{\pi}{2^n}$.

Lemma 2.5. *Let $f : B^2 \to B^2$ be an injection. If f preserves triangles with an interior angle equal to θ, then it is continuous.*

Proof. Let a point $A \in B^2$ and any geodesic ray l starting from A be given. Define the map

$$F : [0, \infty) \to [0, \infty) : t = d(A, X) \mapsto d(A', X'),$$

where the point X runs through l and $d(X, A)$ denotes the hyperbolic distance between the points

X and A. Obviously, F is monotone. Construct a sequence of triangles ABC_n such that for each integer n, $C_n \in l$, $\angle ABC_n = \frac{\pi}{2^n}$ and $\angle BAC_n = \frac{\pi}{2}$. By the hyperbolic Sine and Cosine Law, we have

$$\sinh^2 t_n = \frac{\sinh^2 a \cdot \sin^2\left(\frac{\pi}{2^n}\right)}{1 - \cosh^2 a \cdot \sin^2\left(\frac{\pi}{2^n}\right)},$$

where $a = d(A, B), t_n = d(A, C_n)$. Then $t_n \to 0$ as $n \to \infty$. By the remark after Lemma 2.4, we can deduce $F(t_n) \to 0$ similarly. Hence F is continuous and then the continuity of f at A follows easily. □

Proof of Theorem 1. That f preserves geodesic segments implies f also preserves angles equal to $\pi - \theta$. Hence when $\theta \neq \frac{\pi}{2}$, it follows that f is an isometry by Lemmas 2.4 and 2.5. Now the remaining case is that f preserves right angles. Replacing the quadrilaterals with interior angles $\frac{\pi}{2}, \theta, \theta, \theta$ in the proof of Lemma 2.4 by quadrilaterals with interior angles $\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$. Similarly, we also have f preserves angles equal to $\frac{\pi}{4}$. Inductively, f preserves angles equal to $\frac{\pi}{2^n}$ for each integer n. Thus from the proof of Lemma 2.5, f is continuous and then is Möbius by Lemma 2.2. □

Remark. From the proof of Theorem A in [9], we know that if f preserves Lambert quadrilaterals, then it also preserves right triangles. Then we can show the continuity assumption in Theorem A is irrelevant by the proof of Theorem 1.

Appendix. Classic Möbius transformations are offspring of the theory of functions of one complex variable. They are conformal homeomorphisms of the Riemann sphere identified either with the extended plane $\bar{C}$ or with the 2-sphere $S^2 = \{x \in R^3 \colon |x| = 1\}$. Since Möbius transformations have many beautiful properties in the language of analysis, algebra and geometry, they have several equivalent definitions. Here we just discuss two dimensional Möbius transformations which is the simplest but can mirror many natures of the general case (see [1] for further details).

Let us start from the fractional linear transformations, i.e., those given by

$$f(z) = (az + b)/(cz + d),$$

where $a, b, c, d \in C$ and $ad - bc \neq 0$. All such f form the (orientation-preserving) Möbius group $M\ddot{o}b(2)$.

Recall that the cross ratio of four points $x, y, z, w \in \bar{C}$, where at least three of them are distinct, is defined by

$$|x, y, z, w| = \frac{(x - z)(y - w)}{(x - w)(y - z)}$$

with the understanding that $\frac{x - \infty}{y - \infty} = 1$. Since $f(x) - f(y) = \frac{(x - y)(ad - bc)}{(cx + d)(cy + d)}$, it is immediate that the cross ratio is invariant under all Möbius transformations in the sense that $|x, y, z, w| = |f(x), f(y), f(z), f(w)|$. Furthermore, by [1, Theorem 3.2.7], one can say a map is a Möbius transformation (or a composition of a Möbius transformation and the complex conjugation) if and only if it preserves cross ratios.

The third equivalent definition is: a Möbius transformation is even number of compositions of reflections in circles or lines. This can be seen from that both reflections in circles and lines have the form $f(z) = (a\bar{z} + b)/(c\bar{z} + d)$. Note that this definition plays an important role in the Poincaré extension. Embed $C \cong \{(z, 0)\}$ in $R^3 = \{(z, t)\}$, then each reflection in a circle $S \subset C$ extends to the reflection in the corresponding sphere in R^3 whose equator is S. In an obvious way, each Möbius transformation extends to a (Möbius) map acting on R^3. This is the Poincaré extension. An important feature of the Poincaré extension is that each Möbius transformation acts as an isometry of the hyperbolic 3-space $H^3 = \{(z, t) : t > 0\}$ with the hyperbolic metric $ds = |dx|/t$.

It is also known that a Möbius transformation which preserves the hyperbolic plane $H^2 = \{z = x + iy : y > 0\}$ is an isometry with respect to the hyperbolic metric $ds = |dz|/y$, or equivalently, a Möbius transformation which preserves the hyperbolic disk $B^2 = \{z : |z| < 1\}$ is an isometry with respect to $ds = 2|dz|/(1 - |z|^2)$. In both models, a hyperbolic geodesic is an arc which is orthogonal to the boundary of hyperbolic 2-space.

Recall that the hyperbolic space is an example of non-Euclidean geometry, in particular, the angle sum of a hyperbolic (geodesic) triangle is less than π. Besides the usual Sine and Cosine rules as Euclidean geometry, there is a second Cosine rule in hyperbolic geometry: given a hyperbolic triangle with sides a, b and c and opposite angles α, β and γ, then $\cosh c = \dfrac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}$. Note that this has no analogue in Euclidean geometry. In hyperbolic geometry, this means that if two triangles have the same angles, then there is an isometry mapping one triangle to the other.

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