Stabilization and decay of functionals for linear parabolic control systems

By Takao NAMBU

Department of Applied Mathematics, Faculty of Engineering, Kobe University,
1-1 Rokkodai, Nada, Kobe 657-8501, Japan

(Communicated by Shigefumi MORI, M.J.A., Jan. 15, 2008)

Abstract: We construct a specific feedback control scheme for a class of linear parabolic systems such that some nontrivial linear functionals of the state decay faster than the state, while the state is stabilized. In particular, we raise a new question of pole allocation which is *subject to constraint*, and derive the necessary and sufficient condition: an essential extension of the well known result by W. M. Wonham (1967).

Key words: Linear parabolic systems; feedback stabilization; dynamic compensator.

1. Introduction. Let H be a separable Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. We are interested in the control system with state $u(t) \in H$, $t \geqslant 0$; a finite number of inputs $g_k(t)$, $1 \leqslant k \leqslant M$; and outputs $\langle u(t), w_k\rangle$ with weights $w_k \in H$, $1 \leqslant k \leqslant N$. The control system is described by the linear differential equation in H:

$$\begin{aligned} &\frac{du}{dt} + Lu = \sum_{k=1}^{M} g_k(t) h_k, \quad t > 0, \quad u(0) = u_0, \\ &\quad \text{with outputs} \quad \langle u, w_k\rangle, \quad 1 \leqslant k \leqslant N. \end{aligned} \tag{1}$$

Here, h_k denote actuators, and L a linear closed operator with dense domain $\mathcal{D}(L)$. In stabilization studies, the inputs $g_k(t)$ are designed as a suitable feedback of the outputs $\langle u, w_k\rangle$ –via dynamic compensators (see eqn. (2) below for the precise meaning and setting of the feedback in this paper). The study of feedback stabilization in this scheme has a history of two decades (see the literature, e.g., [2,5,7–9], and [6] for output stabilization), and looks somewhat matured.

Once a decay estimate of $\|u(t)\|$ as $t \to \infty$ is achieved, every linear functional of u clearly decays *at least* with the same decay rate. We then raise a question: *Can we find a nontrivial linear functional which decays faster than* $\|u(t)\|$? The purpose of the paper is to construct a specific feedback control system such that $\|u(t)\|$ decays exponentially with the designated decay rate, while some nontrivial linear functionals of $u(t)$ — viewed as a kind of *outputs* — decay faster than $\|u(t)\|$.

Throughout the paper, the operator L is assumed to be self-adjoint such that the resolvent $(\lambda - L)^{-1}$ is compact. A standard example of such an L is derived from a uniformly elliptic differential operator in a bounded domain in $\mathbf{R}^m$ (see, e.g., [1,3]). According to the Hilbert-Schmidt theory [1,3], there is a set of eigenpairs $(\lambda_i,\ \varphi_{ij})$ satisfying the conditions

(i) $\sigma(L) = \{\lambda_i\}_{i=1}^{\infty}$, $\quad \lambda_1 < \cdots < \lambda_i < \cdots \to \infty$;
(ii) $(\lambda_i - L)\varphi_{ij} = 0$, $i \geqslant 1$, $1 \leqslant j \leqslant m_i (< \infty)$; and
(iii) the set $\{\varphi_{ij}\}$ forms a complete orthonormal system for H. Thus any $u \in H$ is uniquely expressed as a Fourier series: $u = \sum_{i,j} u_{ij}\varphi_{ij}$, $u_{ij} = \langle u, \varphi_{ij}\rangle$. The projector associated with the eigenvalue λ_i is denoted by P_{λ_i}, or $P_{\lambda_i} u = \sum_{j=1}^{m_i} u_{ij}\varphi_{ij}$. Set $P_n = \sum_{i=1}^{n} P_{\lambda_i}$.

The minimum eigenvalue λ_1 is assumed to be negative, so that the system (1) is unstable without any control $g_k(t)$. Let K be the integer such that $\lambda_K \leqslant 0 < \lambda_{K+1}$. In addition the λ_i, $1 \leqslant i \leqslant K$ are assumed to be simple, that is, $m_i = 1$. This is the case, for example, where L is the Sturm-Liouville operator in the bounded interval of $\mathbf{R}^1$. Setting $\varphi_i = \varphi_{i1}$, $1 \leqslant i \leqslant K$, we write the Fourier coefficients $\langle u, \varphi_i\rangle = \langle u, \varphi_{i1}\rangle$ as u_i. The same convension is applied to the vectors w, η, γ, f, and ρ below. Setting $N = 1$ in (1), let us precisely describe our control system: Let $\alpha > 0$ be given, and f belong to $\mathcal{D}(L)$. Our control system has state $(u(t),\ v(t))$, and is described as the system of differential equations in $H \times H$:

2000 Mathematics Subject Classification. Primary 93D15; Secondary 35B35.

$$\frac{du}{dt} + Lu = -\langle v, (\alpha - L)f\rangle\eta + \langle v, \rho\rangle\gamma,$$
$$(2)\qquad \frac{dv}{dt} + Bv = -\langle v, (\alpha - L)f\rangle\eta + \langle v, \rho\rangle\gamma + \langle u, w\rangle h.$$

Here, η, γ, and h denote the actuators in H to be designed; and f the weight generating the functional $\langle u(t), f\rangle$. The equation for v denotes the dynamic compensator, the origin of which is found in [4]. We employ the so-called *identity* compensator, and set

$$(3)\qquad B = L + \langle\cdot, w\rangle h.$$

The operators $-L$ and $-B$ generate analytic semigroups e^{-tL} and e^{-tB}, $t > 0$, respectively. Eqn. (2) is clearly well posed in $H \times H$. The functional $\langle u(t), f\rangle$ is hoped to decay with the decay rate α. The term $\langle v, (\alpha - L)f\rangle\eta$ is introduced for our specific purpose that $\langle u(t), f\rangle$ decay faster than $\|u(t)\|$, and does not appear in regular stabilization schemes.

It is easily seen that $u - v$ satisfies the equation:

$$\frac{d}{dt}(u - v) + B(u - v) = 0, \quad t > 0.$$

Thus we see that $u(t) - v(t) = e^{-tB}(u_0 - v_0)$, $t \geqslant 0$. By assuming the observability conditions:

$$(4)\qquad w_i = \langle w, \varphi_i\rangle \neq 0, \quad 1 \leqslant i \leqslant K,$$

there is an $h \in P_K H$ such that the estimate

$$(5)\qquad \|e^{-tB}\| \leqslant \mathrm{const}\, e^{-\lambda_{K+1}t}, \quad t \geqslant 0$$

holds (see, e.g., [2,7–9]). Here the symbol $\|\cdot\|$ is used for the $\mathcal{L}(H)$-norm, too. Thus we see that

$$\|u(t) - v(t)\| \leqslant \mathrm{const}\, e^{-\lambda_{K+1}t}\|u_0 - v_0\|, \quad t \geqslant 0.$$

Thus the state u is asymptotically identified with v.

The main result of the paper is Theorem 1 below. We just outline the proof since it requires a very long deduction. The complete proof will be reported elsewhere.

2. Main result.

Theorem 1. (i) *Let $0 < \beta < \alpha < \lambda_{K+1}$, and $J \leqslant K$. Suppose that*

$$(6)\qquad w_i \neq 0, \quad 1 \leqslant i \leqslant K, \qquad \eta_i \neq 0, \quad 1 \leqslant i \leqslant J, \quad \text{and}$$

$$\gamma_i \begin{cases} = 0, & 1 \leqslant i \leqslant J, \\ \neq 0, & J < i \leqslant K. \end{cases}$$

Then, we can find the vectors $f = \sum_{1\leqslant i\leqslant J} f_i\varphi_i \in P_J H$ with $\sum_{1\leqslant i\leqslant J} \overline{f_i}\,\eta_i = 1$; $h \in P_K H$; and $\rho \in P_K H$ such that the estimate

$$(7)\qquad \|u(t)\| + \|v(t)\| \leqslant \mathrm{const}\, e^{-\beta t}(\|u_0\| + \|v_0\|), \quad t \geqslant 0$$

holds for every solution $(u(t), v(t))$ to (2). On the other hand, $\langle u(t), f\rangle$ satisfies the decay estimate

$$(8)\qquad |\langle u(t), f\rangle| \leqslant \mathrm{const}\, e^{-\alpha t}, \quad t \geqslant 0.$$

The estimate (7) is no longer improved.

(ii) *Suppose, in addition, that there is an integer $n \geqslant K$ such that*

$$(9)\qquad \langle P_{\lambda_i}\eta, P_{\lambda_i}w\rangle = \langle P_{\lambda_i}\gamma, P_{\lambda_i}w\rangle = 0, \quad i > n.$$

Then the compensator in (2) is reduced to the equation in $\mathbf{C}^{S_n}$, $S_n = \sum_{1\leqslant i\leqslant n} m_i$, with state $v_1(t) = P_n v(t)$. The equation for $(u(t), v_1(t)) \in H \times P_n H$ is described by

$$\frac{du}{dt} + Lu = -\langle v_1, (\alpha - L)f\rangle\eta + \langle v_1, \rho\rangle\gamma,$$
$$(10)\qquad \frac{dv_1}{dt} + B_1 v = -\langle v_1, (\alpha - L)f\rangle P_n\eta + \langle v_1, \rho\rangle P_n\gamma + \langle u, w\rangle h,$$

where B_1 denotes the restriction of B onto the S_n-dimensional subspace $P_n H$: $B_1 = B|_{P_n H} = L|_{P_n H} + \langle\cdot, P_n w\rangle h$. The estimate

$$(11)\qquad \|u(t)\| + \|v_1(t)\| \leqslant \mathrm{const}\, e^{-\beta t}(\|u_0\| + \|v_{10}\|), \quad t \geqslant 0$$

and the decay estimate (8) hold for every solution $(u(t), v_1(t))$ to (10). The estimate (11) is no longer improved. Actually there is a solution such that

$$\|u(t)\| = \mathrm{const}\, e^{-\beta t}, \quad \text{and} \quad \|v_1(t)\| = \mathrm{const}\, e^{-\beta t}, \quad t \geqslant 0.$$

Remark 1. In the general case where $m_i \geqslant 1$, $1 \leqslant i \leqslant K$, an extension of Theorem 1 is possible: We assume N outputs in (1), $N = \max_{1\leqslant i\leqslant K} m_i$.

Remark 2. In the case where $J = K$, the vectors ρ and γ do not appear in (2) and (10). Thus the assumption (9) on γ does not appear, too.

Sketch of the proof (i) We first derive the equation for the functional $\langle u, f\rangle$. In view of the equation for u in (2), we calculate as

$$\frac{d}{dt}\langle u, f\rangle + \langle u, Lf\rangle = -\langle v, (\alpha - L)f\rangle\langle \eta, f\rangle + \langle v, \rho\rangle\langle \gamma, f\rangle.$$

By the assumptions on the vectors f, η, and γ, $\langle u, f\rangle$ satisfies the equation

$$\frac{d}{dt}\langle u, f\rangle + \alpha\langle u, f\rangle = \langle u - v, (\alpha - L)f\rangle. \tag{12}$$

The estimate (8) immediately follows by recalling that $0 < \alpha < \lambda_{K+1}$. The stabilization of the functional $\langle u, f\rangle$ is thus achieved, while the stabilization of the state $u(t)$ is not examined yet at this moment.

Let us establish the decay estimate (7). Set

$$A = L + \langle \cdot, (\alpha - L)f\rangle\eta, \quad \mathcal{D}(A) = \mathcal{D}(L).$$

The adjoint of A is then described by $A^* = L + \langle \cdot, \eta\rangle(\alpha - L)f$, where $\mathcal{D}(A^*) = \mathcal{D}(L)$. Let A_1^* be the restriction of A^* onto the subspace $P_K H$:

$$A_1^* = A^*|_{P_K H} = L_1 + \langle \cdot, P_K\eta\rangle(\alpha - L)f,$$

where $L_1 = L|_{P_K H}$. According to the basis $\{\varphi_i\}_{i=1}^K$ for $P_K H$, we have the following lemma. The proof is straightforward, and thus omitted.

Lemma 2. *The operator A_1^* is identified with the $K \times K$ matrix*:

$$\widehat{A}_1^* = \begin{pmatrix} \Xi & (\alpha - \Lambda_1)\boldsymbol{f}\,\overline{\tilde{\boldsymbol{\eta}}} \\ 0 & \Lambda_2 \end{pmatrix}, \tag{13}$$

where

$$\begin{aligned} \Xi &= \Lambda_1 + (\alpha - \Lambda_1)\boldsymbol{f}\,\overline{\boldsymbol{\eta}}, \\ \Lambda_1 &= \mathrm{diag}\,(\lambda_1\,\lambda_2\,\dots\,\lambda_J), \\ \Lambda_2 &= \mathrm{diag}\,(\lambda_{J+1}\,\lambda_{J+2}\,\dots\,\lambda_K), \\ \boldsymbol{f} &= {}^t(f_1\,f_2\,\dots\,f_J), \\ \boldsymbol{\eta} &= (\eta_1\,\eta_2\,\dots\,\eta_J), \quad \textit{and} \\ \tilde{\boldsymbol{\eta}} &= (\eta_{J+1}\,\eta_{J+2}\,\dots\,\eta_K). \end{aligned} \tag{14}$$

Thus we see that $\sigma(A_1^*) = \sigma(\widehat{A}_1^*) = \sigma(\Xi) \cup \sigma(\Lambda_2)$.

The following theorem discusses on the problem of finite-dimensional pole assignment, and constitutes the key to Theorem 1. We note that the result is essentially different from the well-known pole assignment theory [10] in the sense that it is the problem which is *subject to constraint.*

Theorem 3. *Let Ξ be the $J \times J$ matrix given by* (14)*, and consider its spectrum $\sigma(\Xi)$ which is subject to the constraint*: $\sum_{1 \leqslant i \leqslant J} f_i\,\overline{\eta}_i = 1$. *The number α belongs to $\sigma(\Xi)$ regardless of $\boldsymbol{f}$. For an arbitrary set $\{\mu_i;\ 1 \leqslant i \leqslant J-1\}$ of complex numbers, there is a vector $\boldsymbol{f}$ such that $\sigma(\Xi) = \{\alpha, \mu_1, \mu_2, \dots, \mu_{J-1}\}$, if and only if the observability condition*:

$$\eta_i \neq 0, \quad 1 \leqslant i \leqslant J$$

is satisfied.

Proof of Theorem 3. The spectrum $\sigma(\Xi)$ consists of the solutions to the algebraic equation on λ of order J:

$$\det\left((\alpha - \Lambda_1)^{-1}(\lambda - \Lambda_1) - \boldsymbol{f}\,\overline{\boldsymbol{\eta}}\right) = 0. \tag{15}$$

After calculation, eqn. (15) is rewritten as (except for a constant):

$$0 = (\lambda - \alpha) \times \prod_{2 \leqslant i \leqslant J}(\lambda - \lambda_i)\left(1 + \sum_{2 \leqslant i \leqslant J}\frac{f_i\,\overline{\eta}_i(\lambda_i - \lambda_1)}{\lambda - \lambda_i}\right).$$

Thus α belongs to $\sigma(\Xi)$ regardless of $\boldsymbol{f}$. The necessity part is easy. In fact, if one of the $\eta_1, \dots, \eta_J$ is equal to 0, say $\eta_i = 0$, it is clear that the above equation has the solution $\lambda = \lambda_i$, regardless of the choice of $\boldsymbol{f}$.

The proof of the sufficiency requires a very long deduction. Thus the outline is sketched. The eigenvalues other than α are the solutions to the equation of order $J-1$:

$$0 = \prod_{2 \leqslant i \leqslant J}(\lambda - \lambda_i) + \sum_{2 \leqslant i \leqslant J} f_i\,\overline{\eta}_i(\lambda_i - \lambda_1)\prod_{\substack{2 \leqslant j \leqslant J,\\ j \neq i}}(\lambda - \lambda_j).$$

The first polynomial $\prod_{2 \leqslant i \leqslant J}(\lambda - \lambda_i)$ is a fixed one which we cannot manage. The second one of order $J-2$ is denoted as $F(\lambda)$:

$$\begin{aligned} F(\lambda) &= \sum_{0 \leqslant k \leqslant J-2} A_k\lambda^{J-2-k} \\ &= \sum_{2 \leqslant i \leqslant J} f_i\overline{\eta}_i(\lambda_i - \lambda_1)\left(\lambda^{J-2} + \sum_{1 \leqslant k \leqslant J-2} a_{ik}^J\lambda^{J-2-k}\right). \end{aligned}$$

The problem is then reduced to the problem of an arbitrary allocation of the numbers A_k, $0 \leqslant k \leqslant J-2$. The coefficients a_{ik}^J, and consequently A_k will be described in terms of the λ_i, f_i, and $\overline{\eta}_i$, $1 \leqslant i \leqslant J$. Let σ_j^J, $0 \leqslant j \leqslant J$ be the numbers defined by

$$\sigma_0^J = 1, \quad \text{and}$$

$$\sigma_j^J = \sum_{\substack{1 \leqslant i_1 < i_2 < \\ \cdots < i_j \leqslant J}} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_j}, \quad 1 \leqslant j \leqslant J. \tag{16}$$

For example, $\sigma_1^J = \sum_{1\leqslant i\leqslant J}\lambda_i$. In the above $F(\lambda)$, the $(J-1)$ polynomials: $\lambda^{J-2} + \sum_{1\leqslant k\leqslant J-2} a_{ik}^J \lambda^{J-2-k}$ have similar algebraic structures. The coefficients a_{ik}^J, $1 \leqslant k \leqslant J-2$ are expressed as

$$\begin{aligned}
a_{i1}^J &= -\sigma_1^J + (\lambda_1 + \lambda_i),\\
a_{i2}^J &= \sigma_2^J - \sigma_1^J(\lambda_1+\lambda_i) + (\lambda_1^2 + \lambda_1\lambda_i + \lambda_i^2),\\
a_{i3}^J &= -\sigma_3^J + \sigma_2^J(\lambda_1+\lambda_i) - \sigma_1^J(\lambda_1^2+\lambda_1\lambda_i+\lambda_i^2)\\
&\quad + (\lambda_1^3 + \lambda_1^2\lambda_i + \lambda_1\lambda_i^2 + \lambda_i^3),\\
&\quad \cdots\ \cdots\\
a_{i(J-2)}^J &= (-1)^{J-2}\sigma_{J-2}^J + (-1)^{J-3}\sigma_{J-3}^J(\lambda_1+\lambda_i)\\
&\quad + (-1)^{J-4}\sigma_{J-4}^J(\lambda_1^2+\lambda_1\lambda_i+\lambda_i^2)\\
&\quad + \cdots\\
&\quad - \sigma_1^J(\lambda_1^{J-3} + \lambda_1^{J-4}\lambda_i + \cdots + \lambda_i^{J-3})\\
&\quad + (\lambda_1^{J-2} + \lambda_1^{J-3}\lambda_i + \cdots + \lambda_i^{J-2}),
\end{aligned}$$

for each i, $2 \leqslant i \leqslant J$. By analogy with the a_{ik}^J, we define the number $a_{i(J-1)}^J$ as follows:

$$\begin{aligned}
a_{i(J-1)}^J &= \sum_{k=J-1}^{0} (-1)^k \sigma_k^J \sum_{l=0}^{J-k-1} \lambda_1^{J-k-l-1}\lambda_i^l\\
&= (-1)^{J-1}\sigma_{J-1}^J + (-1)^{J-2}\sigma_{J-2}^J(\lambda_1+\lambda_i)\\
&\quad + (-1)^{J-3}\sigma_{J-3}^J(\lambda_1^2+\lambda_1\lambda_i+\lambda_i^2)\\
&\quad + \cdots\\
&\quad - \sigma_1^J(\lambda_1^{J-2} + \lambda_1^{J-3}\lambda_i + \cdots + \lambda_i^{J-2})\\
&\quad + (\lambda_1^{J-1} + \lambda_1^{J-2}\lambda_i + \cdots + \lambda_i^{J-1}).
\end{aligned}$$

Then we can show the relation: $a_{i(J-1)}^J = 0$, which will be applied to the expression of the A_k. After elementary but tedious calculations, the coefficients A_k, $0 \leqslant j \leqslant J-2$ are finally expressed as

$$\begin{aligned}
A_k &= \sum_{2\leqslant i\leqslant J} f_i\overline{\eta}_i(\lambda_i - \lambda_1)a_{ik}^J\\
&= (-1)^k\sigma_k^J\left(\sum_{1\leqslant i\leqslant J}\lambda_i f_i\overline{\eta}_i - \lambda_1\right)\\
&\quad + (-1)^{k-1}\sigma_{k-1}^J\left(\sum_{1\leqslant i\leqslant J}\lambda_i^2 f_i\overline{\eta}_i - \lambda_1^2\right)\\
&\quad + \cdots - \sigma_1^J\left(\sum_{1\leqslant i\leqslant J}\lambda_i^k f_i\overline{\eta}_i - \lambda_1^k\right)\\
&\quad + \left(\sum_{1\leqslant i\leqslant J}\lambda_i^{k+1} f_i\overline{\eta}_i - \lambda_1^{k+1}\right),
\end{aligned}$$

or in matrix form

$$\begin{pmatrix} A_0\\ A_1\\ A_2\\ \vdots\\ A_{J-3}\\ A_{J-2}\end{pmatrix} = \varSigma \begin{pmatrix} \sum_{1\leqslant i\leqslant J}\lambda_i f_i\overline{\eta}_i\\ \sum_{1\leqslant i\leqslant J}\lambda_i^2 f_i\overline{\eta}_i\\ \sum_{1\leqslant i\leqslant J}\lambda_i^3 f_i\overline{\eta}_i\\ \vdots\\ \vdots\\ \sum_{1\leqslant i\leqslant J}\lambda_i^{J-1} f_i\overline{\eta}_i\end{pmatrix} + \begin{pmatrix} -\lambda_1\\ \sigma_1^J\lambda_1 - \lambda_1^2\\ -\sigma_2^J\lambda_1 + \sigma_1^J\lambda_1^2 - \lambda_1^3\\ \vdots\\ \sum_{i=2}^J(-1)^{J-i}\sigma_{J-i}^J\lambda_1^{i-1}\end{pmatrix}, \tag{17}$$

where $\varSigma$ denotes the nonsingular matrix described by

$$\begin{pmatrix}
1 & 0 & 0 & 0 & \dots & 0 & 0\\
-\sigma_1^J & 1 & 0 & 0 & \dots & 0 & 0\\
\sigma_2^J & -\sigma_1^J & 1 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots\\
(-1)^{J-3}\sigma_{J-3}^J & & & & & 1 & 0\\
(-1)^{J-2}\sigma_{J-2}^J & (-1)^{J-3}\sigma_{J-3}^J & & & \dots & -\sigma_1^J & 1
\end{pmatrix}.$$

Thus the coefficients A_k, $0 \leqslant k \leqslant J-2$ are arbitrarily assigned, if and only if the quantities $B_k = \sum_{1\leqslant i\leqslant J}\lambda_i^k f_i\overline{\eta}_i$, $1 \leqslant k \leqslant J-1$ are freely assigned under the constraint: $\sum_{1\leqslant i\leqslant J} f_i\overline{\eta}_i = 1$. In other words, the A_k are freely assigned, if and only if the equation:

$$\begin{pmatrix}
1 & 1 & \dots & 1\\
\lambda_1 & \lambda_2 & \dots & \lambda_J\\
\lambda_1^2 & \lambda_2^2 & \dots & \lambda_J^2\\
\vdots & \vdots & \ddots & \vdots\\
\lambda_1^{J-1} & \lambda_2^{J-1} & \dots & \lambda_J^{J-1}
\end{pmatrix}
\begin{pmatrix} f_1\overline{\eta}_1\\ f_2\overline{\eta}_2\\ f_3\overline{\eta}_3\\ \vdots\\ f_J\overline{\eta}_J\end{pmatrix}
= \begin{pmatrix} 1\\ B_1\\ B_2\\ \vdots\\ B_{J-1}\end{pmatrix}$$

is solved for any given set of numbers $\{B_1, B_2, \dots, B_{J-1}\}$. However, the equation has a unique solution $\{f_1\overline{\eta}_1, f_2\overline{\eta}_2, \dots, f_J\overline{\eta}_J\}$. Since $\overline{\eta}_i \neq 0$, $1 \leqslant i \leqslant J$, we can determine f_i, $1 \leqslant i \leqslant J$ so that the solutions to $F(\lambda) = 0$ are freely assigned. □

According to Theorem 3, we choose an $\boldsymbol{f}$ such that $\min \sigma(\varXi) \geqslant \beta$, and that the eigenvalues are different from each other: One of the elements of $\sigma(\varXi)$ is, of course, α. Let $\varPi$ be a nonsingular matrix such that

$$\varPi^{-1}\varXi\varPi = \mathrm{diag}\,(\alpha_1\,\alpha_2\,\ldots\,\alpha_J) = \mathcal{A},$$

where $\alpha_1 = \alpha$, and $\alpha_i \geqslant \beta$. For each λ_i, the vector $\begin{pmatrix}\psi_i\\ e_i\end{pmatrix}$ with

$$\psi_i = (\lambda_i - \varXi)^{-1}(\alpha - \varLambda_1)\boldsymbol{f}\,\overline{\tilde{\boldsymbol{\eta}}}\,e_i,$$

$$e_i = {}^t(\overset{J+1)}{0}\ \ldots\ \overset{i)}{1}\ \ldots\ \overset{K)}{0}), \quad J+1 \leqslant i \leqslant K$$

is an eigenvector. Then the nonsingular matrix

$$\varPsi = \begin{pmatrix} \varPi & \psi_{J+1} & \psi_{J+2} & \ldots & \psi_K \\ 0\ldots0 & 1 & 0 & \ldots & 0 \\ 0\ldots0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0\ldots0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

diagonalizes $\widehat{A}_1^*$: $\varPsi^{-1}\widehat{A}_1^*\varPsi = \mathrm{diag}\,(\mathcal{A}\ \varLambda_2)$.

Let us introduce the operator F defined by

$$\begin{aligned} F &= A - \langle\cdot,\rho\rangle\gamma \\ &= L + \langle\cdot,(\alpha - L)f\rangle\eta - \langle\cdot,\rho\rangle\gamma. \end{aligned} \tag{18}$$

The restriction of the adjoint F^* onto the subspace $P_K H$: $F^*|_{P_K H} = A_1^* - \langle\cdot, P_K\gamma\rangle\rho$ is then identified with the matrix

$$\begin{aligned} \widehat{A}_1^* - \boldsymbol{\rho}^*\boldsymbol{\gamma}^* &\longleftrightarrow \varPsi^{-1}(\widehat{A}_1^* - \boldsymbol{\rho}^*\boldsymbol{\gamma}^*)\varPsi \\ &= \begin{pmatrix}\mathcal{A} & 0\\ 0 & \varLambda_2\end{pmatrix} - \varPsi^{-1}\boldsymbol{\rho}^*\ \boldsymbol{\gamma}^*\varPsi, \end{aligned}$$

where $\boldsymbol{\rho}^* = {}^t(\rho_1\,\rho_2\,\ldots\,\rho_K)$, and $\boldsymbol{\gamma}^* = (\gamma_1\ \gamma_2\ \ldots\ \gamma_K)$. By the assumption (6), it is clear that

$$\boldsymbol{\gamma}^*\varPsi = (0\ \ldots\ 0\ \gamma_{J+1}\ \ldots\ \gamma_K) = \boldsymbol{\gamma}^*.$$

By decomposing $\varPsi^{-1}\boldsymbol{\rho}^*$ as $\begin{pmatrix}\boldsymbol{r}_1\\ \boldsymbol{r}_2\end{pmatrix}$, $\boldsymbol{r}_1$: $J\times 1$, and $\boldsymbol{r}_2$: $(K-J)\times 1$, the last matrix is rewritten as

$$\begin{aligned} &\begin{pmatrix}\mathcal{A} & 0\\ 0 & \varLambda_2\end{pmatrix} - \varPsi^{-1}\boldsymbol{\rho}^*\ \boldsymbol{\gamma}^*\varPsi \\ &\quad = \begin{pmatrix}\mathcal{A} & -\boldsymbol{r}_1(\gamma_{J+1}\ \ldots\ \gamma_K)\\ 0 & \varLambda_2 - \boldsymbol{r}_2(\gamma_{J+1}\ \ldots\ \gamma_K)\end{pmatrix}. \end{aligned}$$

Thus we see that

$$\begin{aligned} \sigma\big(F^*|_{P_K H}\big) &= \sigma\big(A_1^* - \langle\cdot, P_K\gamma\rangle\rho\big) \\ &= \{\alpha_1,\,\alpha_2,\,\ldots,\,\alpha_J\} \cup \sigma\big(\varLambda_2 - \boldsymbol{r}_2(\gamma_{J+1}\ \ldots\ \gamma_K)\big). \end{aligned}$$

By the observability assumption: $\gamma_i \neq 0$, $J+1 \leqslant i \leqslant K$, we find a suitable vector $\boldsymbol{r}_2$ such that

$$\min\,\sigma\big(\varLambda_2 - \boldsymbol{r}_2(\gamma_{J+1}\ \ldots\ \gamma_K)\big) = \beta.$$

It is easy to see that β is also the eigenvalue of F^*, and thus $\beta = \overline{\beta}$ is the eigenvalue of F.

The equation for u in (2) is rewritten as

$$\frac{du}{dt} + Fu = \langle u - v, (\alpha - L)f\rangle\eta - \langle u - v, \rho\rangle\gamma,$$

or

$$\begin{aligned} u(t) &= e^{-tF}u_0 \\ &\quad + \int_0^t e^{-(t-s)F}\langle u(s) - v(s), (\alpha - L)f\rangle\eta\,ds \\ &\quad - \int_0^t e^{-(t-s)F}\langle u(s) - v(s), \rho\rangle\gamma\,ds, \end{aligned}$$

from which we establish the estimate:

$$\|u(t)\| \leqslant \mathrm{const}\,e^{-\beta t}, \quad t \geqslant 0;$$

a similar estimate for $v(t)$; and finally the decay estimate (7). This estimate cannot be improved. In fact, let ξ be an eigenvector of F for β:

$$(\beta - F)\xi = 0.$$

By setting $u_0 = v_0 = \xi$, the pair $(u(t),\,v(t)) = (e^{-\beta t}\xi,\,e^{-\beta t}\xi)$ is actually the solution to eqn. (2), and thus the decay estimate (7) is no longer improved.

(ii) We begin with the following proposition.

Proposition 4. *Let p and q be vectors in H, and let $p = \sum_{i,j} p_{ij}\varphi_{ij}$ and $q = \sum_{i,j} q_{ij}\varphi_{ij}$. The function*

$$\langle e^{-tL}Q_n p, Q_n q\rangle, \quad t \geqslant 0$$

is identically equal to 0, if and only if

$$\begin{aligned} &\sum_{l=1}^{m_i} p_{il}\,\overline{q_{il}} = 0, \quad \textit{or} \\ &\langle P_{\lambda_i}p, P_{\lambda_i}q\rangle = 0, \quad i > n. \end{aligned} \tag{19}$$

The idea of the proof is to apply the Laplace transform to the above function and then use analytic continuation of the transformed meromorphic function, but the proof is omitted.

We go back to the equation for v in (2):

$$\begin{aligned} \frac{dv}{dt} + Lv + \langle v, w\rangle h &= -\langle v, (\alpha - L)f\rangle\eta \\ &\quad + \langle v, \rho\rangle\gamma + \langle u, w\rangle h. \end{aligned}$$

Recalling that $f \in P_J H$, $\rho \in P_K H$, and $h \in P_K H$, we divide v into the direct sum:

$$v = v_1 + v_2, \quad v_1 \in P_n H, \quad v_2 \in Q_n H, \quad n \geqslant K.$$

The differential equation for v is then written as the coupling system of equations for v_1 and v_2:

$$
(20) \quad \begin{aligned}
&\frac{dv_1}{dt} + Lv_1 + \langle v_1 + v_2, w\rangle h \\
&\quad = -\langle v_1, (\alpha - L)f\rangle P_n\eta \\
&\qquad + \langle v_1, \rho\rangle P_n\gamma + \langle u, w\rangle h, \\
&\frac{dv_2}{dt} + Lv_2 = -\langle v_1, (\alpha - L)f\rangle Q_n\eta \\
&\qquad\qquad + \langle v_1, \rho\rangle Q_n\gamma.
\end{aligned}
$$

Note that the evolution of v_2 in (20) does not generally disappear, and seriously affects v_1. By the second equation, we see that

$$
\begin{aligned}
v_2(t) &= e^{-tL}Q_n v_0 \\
&\quad - \int_0^t e^{-(t-s)L}\langle v_1(s), (\alpha - L)f\rangle Q_n\eta\, ds \\
&\quad + \int_0^t e^{-(t-s)L}\langle v_1(s), \rho\rangle Q_n\gamma\, ds,
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\langle v_2(t), Q_n w\rangle \\
&\quad = \langle e^{-tL}Q_n v_0, Q_n w\rangle \\
&\qquad - \int_0^t \langle e^{-(t-s)L}Q_n\eta, Q_n w\rangle\langle v_1(s), (\alpha - L)f\rangle ds \\
&\qquad + \int_0^t \langle e^{-(t-s)L}Q_n\gamma, Q_n w\rangle\langle v_1(s), \rho\rangle\, ds.
\end{aligned}
$$

By assumption (9) and Proposition 4 with $p = \eta$ or $= \gamma$ and $q = w$, the second and the third terms disappear. Thus, we see that

$$\langle v_2(t), Q_n w\rangle = \langle e^{-tL}Q_n v_0, Q_n w\rangle, \quad t \geqslant 0.$$

We *choose* the initial data v_0 such that

$$\langle P_{\lambda_i} v_0, P_{\lambda_i} w\rangle = 0, \quad i > n.$$

Then, by Proposition 4 again, we see that

$$\langle v_2(t), Q_n w\rangle = 0. \quad t \geqslant 0.$$

In the equation for v_1 in (20), the term $\langle v_2(t), w\rangle$ then does not appear.

We come to the conclusion: As long as v_0 satisfies $\langle P_{\lambda_i} v_0, P_{\lambda_i} w\rangle = 0$, $i > n$, the new state $(u(t), v_1(t))$ satisfies the system of differential equations in $H \times P_n H$:

$$
(21) \quad \begin{aligned}
&\frac{du}{dt} + Lu = -\langle v_1, (\alpha - L)f\rangle\eta + \langle v_1, \rho\rangle\gamma, \\
&\frac{dv_1}{dt} + Lv_1 + \langle v_1, P_n w\rangle h \\
&\quad = -\langle v_1, (\alpha - L)f\rangle P_n\eta \\
&\qquad + \langle v_1, \rho\rangle P_n\gamma + \langle u, w\rangle h.
\end{aligned}
$$

Eqn. (21), which is nothing but (10), is clearly well posed in $H \times P_n H$, and the decay estimate (11) holds.

To show that (11) is the best possible estimate, we reconsider the eigenvector ξ for the eigenvalue β of F. Actually we obtain—via Proposition 4

Lemma 5. *The eigenvector ξ for the eigenvalue $\beta \in \sigma(F)$ satisfies the relation*:

$$\langle P_{\lambda_i}\xi, P_{\lambda_i} w\rangle = 0, \quad i > n.$$

As we have already seen, the pair: $(u(t), v(t)) = (e^{-\beta t}\xi,\, e^{-\beta t}\xi)$ is a solution to (2), and $v(0) = v_0 = \xi$. Thus, $(u(t),\, v_1(t)) = (e^{-\beta t}\xi,\, e^{-\beta t}\xi_1)$ is a solution to (21), and (11) is the best possible decay estimate.

This finishes the proof of Theorem 1. □

References

[1] S. Agmon, *Lectures on elliptic boundary value problems*, D. Van Nostrand Co., Inc., Princeton, N.J., 1965.

[2] R. F. Curtain, Finite-dimensional compensators for parabolic distributed systems with unbounded control and observation, SIAM J. Control Optim. **22** (1984), no. 2, 255–276.

[3] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, New York, 1983.

[4] D. G. Luenberger, Observers for multivariable systems, IEEE Automatic Control **AC-11** (1966), 190–197.

[5] T. Nambu, On stabilization of partial differential equations of parabolic type: boundary observation and feedback, Funkcial. Ekvac. **28** (1985), no. 3, 267–298.

[6] T. Nambu, Stability enhancement of output for a class of linear parabolic systems, Proc. Roy. Soc. Edinburgh Sec. A **133A** (2003), no. 1, 157–175.

[7] T. Nambu, An $L^2(\Omega)$-based algebraic approach to boundary stabilization for linear parabolic systems, Quart. Appl. Math. **62** (2004), no. 4, 711–748.

[8] T. Nambu, A new algebraic approach to stabilization for boundary control systems of parabolic type, J. Differential Equations **218** (2005), no. 1, 136–158.

[9] Y. Sakawa, Feedback stabilization of linear diffusion systems, SIAM J. Control Optim. **21** (1983), no. 5, 667–676.

[10] W. M. Wonham, On pole assignment in multi-input controllable linear systems, IEEE Trans. Automat. Control **AC-12** (1967), 660–665.