A criterion for topological equivalence of two variable complex analytic function germs

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Abstract: We show that two analytic function germs $(\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ are topologically right equivalent if and only if there is a one-to-one correspondence between the irreducible components of their zero sets that preserves the multiplicites of these components, their Puiseux pairs, and the intersection numbers of any pair of distinct components.

Key words: Topological equivalence of functions; Puiseux pairs; Thom w_f condition; Whitney conditions.

By Zariski [12] and Burau [2], the topological type of an embedded plane curve singularity $(X, 0) \subset (\mathbf{C}^2, 0)$ is determined by the Puiseux pairs of each irreducible component (branch) of this curve and the intersection numbers of any pair of distinct branches. In this note we show the following

Theorem 0.1. *Let f, $g : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be (not necessarily reduced) analytic function germs. Then f and g are topologically right equivalent if and only if there is a one-to-one correspondence between the irreducible components of their zero sets that preserves the multiplicites of these components, their Puiseux pairs, and the intersection numbers of any pair of distinct components.*

Sketch of the proof. The "only if" follows from the above cited result of Zariski and Burau.

To show "if" we proceed as follows: We connect the embedded zero sets $(f^{-1}(0), 0) \subset (\mathbf{C}^2, 0)$ and $(g^{-1}(0), 0) \subset (\mathbf{C}^2, 0)$ by an equisingular (topologically trivial) deformation of plane curve germs

$$(F^{-1}(0), 0 \times P) \subset (\mathbf{C}^2, 0) \times P,$$

where P is a parameter space and $F : (\mathbf{C}^2, 0) \times P \to (\mathbf{C}, 0)$ is analytic. Then, by [14] section 8, the pair $(F^{-1}(0) \setminus 0 \times P, 0 \times P)$ satisfies Whitney conditions. Consequently, by [1,8], the strata $(\mathbf{C}^2, 0) \times P \setminus F^{-1}(0)$, $F^{-1}(0) \setminus 0 \times P$, and $0 \times P$, stratify F as a function with the strong Thom condition w_f. This shows, by Thom-Mather theorem, that F is topologically trivial along P. □

Remark 0.2. The following result was proven in [3] and [9] by different arguments. Let f, $g : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$, $n \geq 2$, be isolated analytic singularities such that the germs $(\mathbf{C}^n, f^{-1}(0), 0)$ and $(\mathbf{C}^n, g^{-1}(0), 0)$ are homeomorphic. Then either f or $\overline{f}$ is topologically right equivalent to g.

Remark 0.3. In the isolated singularity case we may proceed as follows: By [15] and [7], the embedded topological type of a plane curve singularity at a singular point is determined by the embedded topological type of each analytically irreductible component and the intersection of any pair of distinct branches. Using this result it was shown in [6] that the multiplicity is a topological invariant of plane curve singularity and that a μ constant deformation of two variable isolated singularity germ is topologically trivial.

Note also that by Zariski [13] an equisingular family of plane curves is μ constant and multipliciy constant, i.e. μ^* constant in the sense of Teissier [10]. It is known, c.f. [1,8], and the bibliography quoted therein, that a μ^* constant family of isolated singularities satisfies the strong Thom condition w_f.

Remark 0.4. In [5] Kuo and Lu introduced a tree model $T(f)$ of an isolated singularity $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$. This model allows one to visualise the Puiseux pairs of irreducible components of $f^{-1}(0)$ and the contact orders between them. Kuo and Lu's model can be easily addapted to the non-isolated case by adding the multiplicities of components. Then theorem 0.1 says that f and g are

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topologically right equivalent if and only if their tree models coincide.

We give now details. First we connect f by an equisingular deformation to a normal family that depends only on the embedded topological type of $(f^{-1}(0),0) \subset (\mathbf{C}^2,0)$ and the multiplicities of its branches. A similar construction gives an equisingular deformation connecting f and g. For these deformations we give elementary proofs of Whitney and strong Thom conditions and explicit formulae for vector fields trivializing them.

0.1. Deformation of f to a normal family. Fix $f : (\mathbf{C}^2,0) \to (\mathbf{C},0)$. Choose a system of coordinates so that $y = 0$ is transverse to the tangent cone to $f = 0$ at the origin. Write

$$f(x,y) = \prod_{k=1}^{N} (f_k(x,y))^{d_k} \tag{0.1}$$

$$= u(x,y) \prod_{k=1}^{N} \prod_{j=1}^{m_k} (x - \lambda_{k,j}(y))^{d_k},$$

where $f_1,\dots,f_N$ are irreducible germs, $u(x,y)$ is a unit, $u(0,0) \neq 0$, and $x = \lambda_{k,i}(y)$ are Newton-Puiseux roots of f_k. Then

$$\lambda_{k,i}(y) = \sum a_\alpha(k,i) y^\alpha,$$

are fractional power series: $\alpha \in \frac{1}{m_k}\mathbf{N}$. The coefficients $a_\alpha(k,i)$ are well-defined if we restrict λ to a real half-line through the origin. In what follows we choose $y \in \mathbf{R}, y \geq 0$. This allows us to define *the contact order* between two such roots as

$$O(\lambda_{k_1,i}, \lambda_{k_2,j}) := \operatorname{ord}_0(\lambda_{k_1,i} - \lambda_{k_2,j})(y),$$

$y \in \mathbf{R}, y \geq 0$. All roots of f_k can be obtained from $\lambda_{k,1}$ by

$$\lambda_{k,j}(y) = \sum a_\alpha(k,1)\theta^{(j-1)\alpha m_k} y^\alpha,$$

where $\theta = e^{2\pi i/m_k}$.

Denote by $\Lambda_{all,k} = \{\alpha_j\}$ the set of all contact orders between $\lambda_{k,1}(y)$ and the other Newton-Puiseux roots of f. The Puiseux exponents of $\lambda_{k,j}(y)$ form a subset $\Lambda_{P,k} \subset \Lambda_{all,k}$. Clearly $a_\alpha(k,j) \neq 0$ if $\alpha \in \Lambda_{P,k}$. The other exponents of $\Lambda_{all,k}$ can be divided into two groups. If the denominator of α does not divide the greatest common multiple of the denominators of $\alpha' \in \Lambda_{P,k}$, $\alpha' < \alpha$, then $a_\alpha(k,j) = 0$, since otherwise $\alpha \in \Lambda_{P,k}$. For the remaining exponents there is no condition on the coefficient $a_\alpha(k,j)$, so we denote their set by $\Lambda_{free,k}$. Finally we set

$$\Lambda_k := \Lambda_{P,k} \cup \Lambda_{free,k}.$$

For fixed k we order the roots $\lambda_{k,j}, j = 1,\dots, m_k$, by the lexicographic order on the sequences $(\arg(a_\alpha(k,j)), \alpha \in \Lambda_{P,k})$. Here $\arg \in [0, 2\pi)$. We obtain exactly the same ordering if we use the lexicographic order on $(\arg(a_\alpha(k,j)), \alpha \in \Lambda_k$, $a_\alpha(k,j) \neq 0)$, since for two conjugate roots $O(\lambda_{k,i}, \lambda_{k,j})$ is a Puiseux exponent. At the beginning the choice of $\lambda_{k,1}$ was arbitrary. From now on we assume that $\lambda_{k,1}$ is the smallest among all $\lambda_{k,i}$ with respect to the above lexicographic order.

Lemma 0.5. *Let $k_1 \neq k_2$. Then*

$$\max_{i,j} O(\lambda_{k_1,i}, \lambda_{k_2,j}) = O(\lambda_{k_1,1}, \lambda_{k_2,1}).$$

Proof. Suppose, contrary to our claim, that

$$O(\lambda_{k_1,1}, \lambda_{k_2,1}) < O(\lambda_{k_1,i}, \lambda_{k_2,j}).$$

Then there exist j' and i' such that

$$O(\lambda_{k_1,i}, \lambda_{k_2,j}) = O(\lambda_{k_1,1}, \lambda_{k_2,j'}) = O(\lambda_{k_1,i'}, \lambda_{k_2,1}).$$

This shows that for $\alpha = O(\lambda_{k_1,1}, \lambda_{k_2,1})$, $a_\alpha(k_1,1) = a_\alpha(k_2,j') \neq 0$ and hence

$$\arg(a_\alpha(k_1,1)) = \arg(a_\alpha(k_2,j')) > \arg(a_\alpha(k_2,1)).$$

Similarly

$$\arg(a_\alpha(k_2,1)) = \arg(a_\alpha(k_1,i')) > \arg(a_\alpha(k_1,1))$$

that is impossible. □

Definition 0.6. We define *the deformation space* $D(f) \subset \prod_k \mathbf{C}^{|\Lambda_k|}$ as follows: Write an element of $\mathbf{C}^{|\Lambda_k|}$ as $a(k) = (a_\alpha(k); \alpha \in \Lambda_k) \in \mathbf{C}^{|\Lambda_k|}$. Then, $\underline{a} = (a(k); k = 1,\dots,N) \in D(f)$ if :

(1) if $\alpha \in \Lambda_{P,k}$ the $a_\alpha(k) \neq 0$,
(2) if $\alpha < O(\lambda_{k_1,1}, \lambda_{k_2,1})$ and $\alpha \in \Lambda_{k_1} \cap \Lambda_{k_2}$ then $a_\alpha(k_1) = a_\alpha(k_2)$.
(3) if $\alpha = O(\lambda_{k_1,1}, \lambda_{k_2,1})$ and $\alpha \in \Lambda_{k_1} \cap \Lambda_{k_2}$ then $a_\alpha(k_1) \neq a_\alpha(k_2)$.

We write each Newton-Puiseux root of f as

$$\lambda_{k,j}(y) = \sum_{\alpha \geq 1} a_\alpha(k,j) y^\alpha = \sum_{\alpha \in \Lambda_k} a_\alpha(k,j) y^\alpha + R_{k,j}(y), \tag{0.2}$$

and then for $(s, \underline{a}) \in \mathbf{C} \times D(f)$

$$\lambda_{k,j}(s, \underline{a}, y) = \sum_{\alpha \in \Lambda_k} a_\alpha(k) \theta_k^{(j-1)\alpha m_k} y^\alpha + s R_{k,j}(y),$$

where $\theta_k = e^{2\pi i/m_k}$. Consider the following deformation of f

(0.3)
$$F(\tau,s,u_0,\underline{a},x,y)=$$
$$(\tau(u(x,y)-u(0,0))+u_0)\prod_{k=1}^{N}\prod_{i=1}^{m_k}(x-\lambda_{k,i}(s,\underline{a},y))^{d_k},$$
where $(\tau,s,u_0,\underline{a})\in P:=\mathbf{C}\times\mathbf{C}\times\mathbf{C}^*\times D(f)$. ($P$ stands for the parameter space). Then F is analytic in all variables and $f(x,y)=F(1,1,u(0,0),\underline{a}(f),x,y)$, where $\underline{a}(f)$ is given by the coefficients $a_\alpha(k,1)$ of the Newton-Puiseux roots of f.

We call $F(0,0,1,\underline{a},x,y):D(f)\times(\mathbf{C}^2,0)\to(\mathbf{C},0)$ *the normal family of germs associated to* f. It depends only on the topological type of $(f^{-1}(0),0)\subset(\mathbf{C}^2,0)$ and the multiplicities d_k of irreducible components of f.

We may as well embedd f and g in one equisingular family by taking

(0.4) $$F(\tau,s,u_0,\underline{a},x,y)=$$
$$(\tau_1u_1+\tau_2u_2+u_0)\prod_{k=1}^{N}\prod_{i=1}^{m_k}(x-\lambda_{k,i}(s,\underline{a},y))^{d_k},$$
where $u_1(x,y)=u_f(x,y)-u_f(0,0)$ and $u_2(x,y)=u_g(x,y)-u_g(0,0)$ and u_f, resp. u_g, denote the unit of (0.1) for f and g respectively, $s=(s_f,s_g)\in\mathbf{C}^2$ and
$$\lambda_{k,j}(s,\underline{a},y)=\sum_{\alpha\in\Lambda_k}a_\alpha(k)\theta_k^{(j-1)\alpha m_k}y^\alpha+s_fR_{f,k,j}+s_gR_{g,k,j}.$$

0.2. Whitney Conditions and Thom Condition. Let $X=F^{-1}(0)\subset U\subset P\times\mathbf{C}^2$, where U is a small open neighbourhood of $P\times\{0,0\}$ in $P\times\mathbf{C}^2$. We show by elementary computations that $(U\setminus X,X\setminus P,P)$ as a stratification, satisfies Whitney conditions and the strong Thom condtion w_f.

Let G be a reduced equation of X:
$$G(\tau,s,\underline{a},x,y)=u(\tau,x,y)\prod_{k=1}^{N}\prod_{i=1}^{m_k}(x-\lambda_{k,i}(s,\underline{a},y)),$$
The pair $(X\setminus P,P)$ satisfies Verdier condition w, [11], equivalent to Whitney conditions in the complex analytic case, if for any $(x_0,y_0)\in P$ there is a constant C such that on X near (x_0,y_0) the following inequality holds

(0.5) $$\left|\frac{\partial G}{\partial v}\right|\le C\|(x,y)\|\left\|\left(\frac{\partial G}{\partial x},\frac{\partial G}{\partial y}\right)\right\|$$

where v denotes any of the coordinates on the parameter space P. A direct computation on $x=\lambda_{k_0,i_0}(s,\underline{a},y)$ gives
$$\frac{\partial G}{\partial v}=\left(-\frac{\partial\lambda_{k_0,i_0}}{\partial v}\right)\prod_{(k,i)\neq(k_0,i_0)}(x-\lambda_{k,i}(s,\underline{a},y)),$$
$$\frac{\partial G}{\partial x}=\prod_{(k,i)\neq(k_0,i_0)}(x-\lambda_{k,i}(s,\underline{a},y)).$$
and (0.5) follows from

(0.6) $$\left|\frac{\partial\lambda_{k_0,i_0}}{\partial v}\right|\le C|y|,$$

that is a consequence of the fact that all exponents in $\bigcup_k\Lambda_k$ are ≥ 1.

Similarly, $(U\setminus X,X\setminus P,P)$ as a stratification of F satisfies the strong Thom condition w_f if for any $(x_0,y_0)\in P$ there is a constant C such that in a neighborhood of (x_0,y_0) the following inequality holds

(0.7) $$\left|\frac{\partial F}{\partial v}\right|\le C\|(x,y)\|\left\|\left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y}\right)\right\|$$

in the complement of the zero set of F.

Replace y by t^n so that the roots $\lambda_{k,i}(s,\underline{a},t^n)$ become analytic. We denote them using a single index as $\tilde\lambda_m(s,\underline{a},t)$. Given a power series $\xi(v,t)=\sum_{\alpha=1}^\infty a_\alpha(v)t^\alpha$, where $v\in P$, we consider *a horn-neighborhood of* ξ
$$H_d(\xi,w)=\{(x,v,t);|x-\xi(v,t)|\le w|t|^d\},$$
where $d\in\mathbf{N}\cup\{0\}$ and $w>0$, compare [5] §6. In $H_d(\xi,w)$ we use the coordinates $\tilde x,t,v$, where
$$x=\tilde xt^d+\xi(v,t).$$
In what follows $\xi(v,t)$ will be one of the roots $\tilde\lambda_m$.

Denote $e_{m,j}=O(\tilde\lambda_m,\tilde\lambda_j)$ for $j\neq m$. Let $C>0$ stand for a large constant and $\varepsilon>0$ for a small constant.

Lemma 0.7. *For any* $d\in\mathbf{N}\cup\{0\}$, (0.7) *holds on* $\hat H_d(\tilde\lambda_m,\varepsilon,C):=H_d(\tilde\lambda_m,\varepsilon)\setminus\bigcup_l H_{e_{m,j}}(\tilde\lambda_m,C)$, *where the union is taken over all* $j\neq m$ *such that* $e_{m,j}>d$.

Proof. Denote $I=\{j;\,e_{m,j}>d\}$, $m\in I$. Then on $\hat H_d(\tilde\lambda_m,\varepsilon,C)$
$$F(v,\tilde x,t)=u(v,\tilde x,t)\,t^M\prod_{j\in I}(\tilde x-\beta_j(v,t))^{d_j},$$
where $M=\sum_{j\notin I}d_je_{m,j}+d\sum_{j\in I}d_j$, u is a unit, and $\beta_j=t^{-d}(\tilde\lambda_j-\tilde\lambda_m)$, $\beta_m\equiv 0$. Then

(0.8) $$\frac{\partial F}{\partial v}=\left(\sum_{j\in I}d_k\frac{-\partial\beta_j/\partial v}{\tilde x-\beta_j(t,v)}+\frac{\partial u/\partial v}{u}\right)F,$$

$$(0.9)\quad \frac{\partial F}{\partial x} = t^{-d}\frac{\partial F}{\partial \tilde{x}} = \left(\sum_{j\in I} d_k \frac{1}{\tilde{x}-\beta_j(t,v)} + \frac{\partial u/\partial \tilde{x}}{u}\right) t^{-d} F.$$

If $(\tilde{x},t,v)\in \hat{H}_d(\tilde{\lambda}_m,\varepsilon,C)$ and $j\in I, j\neq m$, then $|\beta_j|\le \delta|\tilde{x}|$, for δ small provided C is chosen large. Hence $|\tilde{x}-\beta_j|\sim|\tilde{x}|$. Moreover, for ε small, the first summand in the parenthesis of (0.9) is dominant. Consequently for any $j\in I$ and $(\tilde{x},t,v)\in \hat{H}_d(\tilde{\lambda}_m,\varepsilon,C)$

$$\left|\frac{\partial F}{\partial \tilde{x}}\right| \ge \frac{c}{|\tilde{x}-\beta_j|}\,|F|,$$

where c stands for a small constant. Then (0.7) follows from (0.8) and (0.6) since $|\frac{\partial \beta_j}{\partial v}|\le C|t^{-d}y|$.

The case $I=\{m\}$ is left to the reader. □

Lemma 0.8. *Fix $d=e_{m,j}$. Then (0.7) holds on $\tilde{H}_d(\tilde{\lambda}_m,C,\varepsilon)=H_d(\tilde{\lambda}_m,C)\setminus\bigcup_l H_d(\tilde{\lambda}_l,\varepsilon)$, where the union is taken over all l such that $O(\tilde{\lambda}_m,\tilde{\lambda}_l)\ge d$ (including $l=m$).*

Proof. On $\tilde{H}_d(\tilde{\lambda}_m,C,\varepsilon)$, $F(v,\tilde{x},t)=u(v,\tilde{x},t)\,t^M$, and hence

$$\left|t\frac{\partial F}{\partial t}\right| = \left|\left(M+\frac{t\partial u/\partial t}{u}\right)F\right| \ge (M-\delta)\cdot|F|$$

where $\delta\to 0$ as $t\to 0$. Then by a direct computation of $\partial F/\partial t$

$$\left|\frac{\partial F}{\partial v}\right| \le C_1|F| \le C_2\left|t\frac{\partial F}{\partial t}\right| \le C_3|y|\left\|\left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y}\right)\right\|$$

□

To complete the proof of (0.7) we note that a finite family of horns $\hat{H}_d(\tilde{\lambda}_m,\varepsilon,C)$, $\tilde{H}_d(\tilde{\lambda}_m,C,\varepsilon)$, with $d=e_{m,j}$ or $d=0$, covers a neighborhood of the origin in $\mathbf{C}^2$ (times a neighborhood in the parameter space).

0.3. Trivialization. By Thom-Mather theory the family F is locally topologically trivial along P. Moreover, it can be show easily that it can be trivialized by Kuo's vector field, cf. [4]. Suppose that the parameter P space is one-dimensional with the parameter v. Define the vector field V on each stratum separately by formulae :

$$\frac{\partial}{\partial v} \qquad \text{on } P$$

$$\frac{\partial}{\partial v} - \frac{\partial G/\partial v}{\|\operatorname{grad}_{x,y} G\|^2}\left(\frac{\overline{\partial G}}{\partial x}\frac{\partial}{\partial x}+\frac{\overline{\partial G}}{\partial y}\frac{\partial}{\partial y}\right) \quad \text{on } X\setminus P$$

$$\frac{\partial}{\partial v} - \frac{\partial F/\partial v}{\|\operatorname{grad}_{x,y} F\|^2}\left(\frac{\overline{\partial F}}{\partial x}\frac{\partial}{\partial x}+\frac{\overline{\partial F}}{\partial y}\frac{\partial}{\partial y}\right) \quad \text{on } U\setminus X.$$

The flow of V is continuous and preserves the levels of F by the arguments of [4].

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