# A rationality problem of some Cremona transformation

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**Abstract:** In this note we give a new approach to the rationality problem of some Cremona transformation. Let $k$ be any field, $k(x,y)$ be the rational function field of two variables over $k$. Let $\sigma$ be a $k$-automorphism of $k(x,y)$ defined by

$$\sigma(x) = \frac{-x(3x-9y-y^2)^3}{(27x+2x^2+9xy+2xy^2-y^3)^2}, \quad \sigma(y) = \frac{-(3x+y^2)(3x-9y-y^2)}{27x+2x^2+9xy+2xy^2-y^3}.$$

Theorem. The fixed field $k(x,y)^{\langle\sigma\rangle}$ is rational (= purely transcendental) over $k$. Embodied in a new proof of the above theorem are several general guidelines for solving the rationality problem of Cremona transformations, which may be applied elsewhere.

**Key words:** Rationality problem; Cremona transformations; linear actions; monomial group actions.

**1. Introduction.** Let $k$ be any field, $k(x_1,\ldots,x_n)$ be the rational function field of $n$ variables. (It is not necessary to assume that $k$ is algebraically closed.) By a Cremona transformation on $\mathbf{P}^n$ we mean a $k$-automorphism $\sigma$ on $k(x_1,\ldots,x_n)$, i.e.

$$(1) \qquad \sigma : k(x_1,\ldots,x_n) \longrightarrow k(x_1,\ldots,x_n)$$

where $\sigma(x_i) \in k(x_1,\ldots,x_n)$ for each $1 \le i \le n$ and $\sigma$ is an automorphism. We will denote by $\mathrm{Cr}_n$ the group of all Cremona transformations on $\mathbf{P}^n$. The purpose of this note is to consider whether $k(x_1,x_2)^G$ is rational (= purely transcendental) over $k$ where $G$ is some finite subgroup of $\mathrm{Cr}_2$.

Note that, if $k$ is algebraically closed, then $k(x_1,x_2)^G$ is rational over $k$ by Zariski-Castelnuovo's Theorem [Za]. On the other hand, if the group $G$ consists of automorphisms $\sigma$ such that, in (1), $\sigma(x_i)$ are homogeneous linear polynomials (resp. monomials) in $x_1,\ldots,x_n$, then the group action of $G$ on $k(x_1,\ldots,x_n)$ is the usual linear action (resp. the monomial group action). The rationality problem of linear actions or the monomial group actions has been investigated extensively. See, for examples, [Sw,KP,HK1,HK2,HR]. It seems that not many research works are devoted to the rationality problem of "genuine" Cremona transformations, i.e. the $\sigma(x_i)$ in (1) are, instead of linear polynomials or monomials, rational functions with total degrees high enough, say, $\ge 4$. As far as we know, only ad hoc techniques can be found in the literature for solving the rationality problems of Cremona transformations.

In this note we give a new approach to the rationality problem of some Cremona transformation. We show the following theorem which was given in [HM]***) from the view point of Tschirnhausen transformations of cubic generic polynomials.

**Theorem 1** ([HM] Theorem 10). *Let $k$ be any field and $k(x_1,x_2)$ be the rational function field of two variables over $k$. Let $\sigma \in \mathrm{Cr}_2$ defined by*

$$\sigma : k(x_1,x_2) \longrightarrow k(x_1,x_2)$$

*where*

$$\sigma(x_1) = \frac{-x_1(3x_1-9x_2-x_2^2)^3}{(27x_1+2x_1^2+9x_1x_2+2x_1x_2^2-x_2^3)^2},$$
$$\sigma(x_2) = \frac{-(3x_1+x_2^2)(3x_1-9x_2-x_2^2)}{27x_1+2x_1^2+9x_1x_2+2x_1x_2^2-x_2^3}.$$

*Then $k(x_1,x_2)^{\langle\sigma\rangle} := \{f \in k(x_1,x_2) \ : \sigma(f) = f\}$ is rational over $k$.*

Note that $\sigma^2 = 1$.

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***) A note by the first-named author: in the proof in [HM, p.25] unnecessary two maps $\sigma_1, \sigma_2$ should be deleted.

Many rationality problems arise from the study of moduli spaces of some geometric configurations. The rationality problem in Theorem 1 arose in the study of the moduli of cubic generic polynomials. See [HM].

We will give a new proof of Theorem 1 in Section 2 (when $\operatorname{char} k \neq 2,3$) and Section 3 (when $\operatorname{char} k = 2$, or 3). Our proof is completely different from that in [HM]. We hope that this proof will be helpful to people working on the rationality problem of Cremona transformations because it contains systematic methods for attacking the rationality problem. (See Step 1, Step 2 and Step 5 of Section 2, in particular.) In keeping with the spirit of the proof in Section 2 we give another proof of the case $\operatorname{char} k = 2$ and the case $\operatorname{char} k = 3$ in Section 4 and Section 5 respectively.

Some symbolic computations in this note are carried out with the aid of "Mathematica" [Wo].

Finally we will emphasize that it is unnecessary to assume that the base field $k$ is algebraically closed or any restriction on the characteristic of $k$.

**2. The case $\operatorname{char} k \neq 2, 3$.** Throughout this section, we assume that $\operatorname{char} k \neq 2,3$.

**Step 1.** Note that $\sigma$ induces a birational map on $\mathbf{P}^2$. We will find some irreducible exceptional divisors of this rational map. Clearly the curve defined by $3x_1 - 9x_2 - x_2^2 = 0$ is one of the candidates. Taking its image $\sigma(3x_1 - 9x_2 - x_2^2)$, we will find another polynomial. Thus, define

$$\begin{aligned} y_1 &= 3x_1 - 9x_2 - x_2^2, \\ y_2 &= 27x_1 + 9x_1x_2 + x_2^3, \\ y_3 &= -27x_1 - 2x_1^2 - 9x_1x_2 - 2x_1x_2^2 + x_2^3. \end{aligned} \tag{2}$$

With the aid of computers, it is easy to see that

$$\sigma \ : \ \begin{aligned} y_1 &\longmapsto y_1y_2^2y_3^{-2}, \\ y_2 &\longmapsto y_1^3y_2^2y_3^{-3}, \\ y_3 &\longmapsto y_1^3y_2^3y_3^{-4}. \end{aligned} \tag{3}$$

Note that the determinant of the exponents of the above map is

$$\det\begin{pmatrix} 1 & 3 & 3 \\ 2 & 2 & 3 \\ -2 & -3 & -4 \end{pmatrix} = 1.$$

Thus the action of $\sigma$ on $k(y_1, y_2, y_3)$ can be lifted to $k(Y_1, Y_2, Y_3)$ ($Y_1, Y_2, Y_3$ are algebraically independent over $k$) and induces a monomial action on $k(Y_1, Y_2, Y_3)$. But we will not use this fact in the following steps.

**Step 2.** Luckily we find that $k(y_1, y_2, y_3) = k(x_1, x_2)$. In fact, from (2), we may eliminate $x_2$ and get two polynomial equations of $x_1$ with coefficients in $k(y_1, y_2, y_3)$; applying the Euclidean algorithm to these two polynomials, we may show that $x_1 \in k(y_1, y_2, y_3)$.

More explicitly, with the aid of computers, we will find (i) the expressions of $x_1, x_2$ in terms of $y_1, y_2, y_3$, and (ii) a polynomial equations of $y_1, y_2, y_3$. We get

$$x_1 = \frac{-2y_1^3 - 729y_2 + 27y_1y_2 - 2y_2^2 - 729y_3 + 27y_1y_3}{108(y_2 + y_3)},$$

$$x_2 = \frac{-2y_1^4 + 9y_1^2y_2 - 2y_1y_2^2 + 9y_1^2y_3 + 81y_2y_3 + 81y_3^2}{18(y_1^3 - y_2y_3)},$$

$$\begin{aligned} f(y_1, y_2, y_3) = {} & 2y_1^6 + 729y_1^3y_2 - 27y_1^4y_2 \\ & + 4y_1^3y_2^2 - 27y_1y_2^3 + 2y_2^4 + 729y_1^3y_3 - 27y_1^4y_3 \\ & - 27y_1y_2^2y_3 + 729y_2y_3^2 + 729y_3^3 = 0. \end{aligned} \tag{4}$$

**Step 3.** The map of $\sigma$ defined in (3) can be simplified as follows: Define

$$z_1 = y_2^{-1}y_3, \quad z_2 = y_1y_2^{-1}, \quad z_3 = y_1^{-2}y_3.$$

It follows that $k(y_1, y_2, y_3) = k(z_1, z_2, z_3)$ and

$$\sigma \ : \ z_1 \longmapsto 1/z_1, \quad z_2 \longmapsto z_3 \longmapsto z_2. \tag{5}$$

The relation $f(y_1, y_2, y_3) = 0$ in (4) becomes

$$\begin{aligned} g(z_1, z_2, z_3) = {} & 2z_1^2z_2^2 + 4z_1z_2z_3 - 27z_1z_2^2z_3 \\ & - 27z_1^2z_2^2z_3 + 2z_3^2 - 27z_2z_3^2 - 27z_1z_2z_3^2 + 729z_2^3z_3^2 \\ & + 729z_1z_2^3z_3^2 + 729z_1z_2^2z_3^3 + 729z_1^2z_2^2z_3^3 = 0. \end{aligned} \tag{6}$$

**Step 4.** The map of $\sigma$ defined in (5) is equivalent to

$$\begin{aligned} \sigma \ : \ & z_2 - z_3 \longmapsto -(z_2 - z_3), \\ & \frac{1 - z_1}{1 + z_1} \longmapsto -\frac{1 - z_1}{1 + z_1}, \quad z_2 + z_3 \longmapsto z_2 + z_3. \end{aligned}$$

Thus $k(x_1, x_2)^{\langle\sigma\rangle} = k(z_1, z_2, z_3)^{\langle\sigma\rangle} = k(u_1, u_2, u_3)$ where $u_1, u_2, u_3$ are defined by

$$u_1 = (z_2 - z_3)^2, \quad u_2 = \left(\frac{1 - z_1}{1 + z_1}\right) \cdot (z_2 - z_3),$$

$$u_3 = z_2 + z_3.$$

The relation $g(z_1, z_2, z_3) = 0$ in (6) becomes

$$(7)\quad 108u_1u_2 - 729u_1^2u_2 - 16u_2^2 - 108u_1u_3 - 729u_1^2u_3 + 32u_2u_3 - 16u_3^2 - 108u_2u_3^2 + 1458u_1u_2u_3^2 + 108u_3^3 + 1458u_1u_3^3 - 729u_2u_3^4 - 729u_3^5 = 0.$$

In conclusion, $k(x_1, x_2)^{\langle\sigma\rangle}$ is a field generated by $u_1, u_2, u_3$ over $k$ with the relation (7). We will simplify the relation (7) to get two generators.

**Step 5.** The relation (7) defines an algebraic surface. However this algebraic surface contains singularities. We will make some change of variables to simplify the singularities and the equation (7). Define

$$v_1 = u_1u_3^{-1}, \quad v_2 = u_2u_3^{-1}, \quad v_3 = u_3.$$

Then $k(u_1, u_2, u_3) = k(v_1, v_2, v_3)$ and the relation (7) becomes

$$(8)\quad h(v_1, v_2, v_3) = 16 + 108v_1 - 32v_2 - 108v_1v_2 + 16v_2^2 - 108v_3 + 729v_1^2v_3 + 108v_2v_3 + 729v_1^2v_2v_3 - 1458v_1v_3^2 - 1458v_1v_2v_3^2 + 729v_3^3 + 729v_2v_3^3 = 0.$$

We will determine the singularities of $h(v_1, v_2, v_3) = 0$ by solving

$$h(v_1, v_2, v_3) = \frac{\partial h}{\partial v_i}(v_1, v_2, v_3) = 0$$

for $i = 1, 2, 3$. Then we get $v_2 - 1 = v_1 - v_3 = 0$. Define

$$w_1 = v_1 - v_3, \quad w_2 = v_2 - 1, \quad w_3 = v_3.$$

Therefore we have $k(v_1, v_2, v_3) = k(w_1, w_2, w_3)$ and the relation (8) becomes

$$108w_1w_2 - 16w_2^2 - 1458w_1^2w_3 - 729w_1^2w_2w_3 = 0.$$

The above equation is a linear equation in $w_3$. Thus $w_3 \in k(w_1, w_2)$. It follows $k(w_1, w_2, w_3) = k(w_1, w_2)$. We conclude that $k(x_1, x_2)^{\langle\sigma\rangle} = k(w_1, w_2, w_3) = k(w_1, w_2)$ is rational over $k$.

**Step 6.** We will give explicit formulae of $w_1, w_2$ in terms of $x_1, x_2$. It is not difficult to find that

$$\frac{w_1}{w_2} = \frac{-4(3x_1 - 9x_2 - x_2^2)}{27(27 + x_1 + 9x_2 + x_2^2)},$$
$$w_2 = \frac{27(27x_1 + 2x_1^2 + 9x_1x_2 + 2x_1x_2^2 - x_2^3)}{27x_1^2 + 18x_1^2x_2 - 27x_1x_2^2 + 27x_2^3 + 2x_1x_2^3}.$$

Finally we obtain

$$k(x_1, x_2)^{\langle\sigma\rangle} = k\left(\frac{3x_1 - 9x_2 - x_2^2}{27 + x_1 + 9x_2 + x_2^2}, \frac{27x_1 + 2x_1^2 + 9x_1x_2 + 2x_1x_2^2 - x_2^3}{27x_1^2 + 18x_1^2x_2 - 27x_1x_2^2 + 27x_2^3 + 2x_1x_2^3}\right).$$

## 3. The remaining cases.

**Step 1.** In this step, we assume that $\operatorname{char} k = 2$. Note that the automorphism $\sigma$ becomes

$$x_1 \longmapsto \frac{x_1(x_1 + x_2 + x_2^2)^3}{(x_1^2 + x_1x_2 + x_2^3)^2},$$
$$x_2 \longmapsto \frac{(x_1 + x_2^2)(x_1 + x_2 + x_2^2)}{x_1^2 + x_1x_2 + x_2^3}.$$

Define

$$y_1 = x_1 + x_2 + x_2^2, \quad y_2 = x_2.$$

Then we have $k(x_1, x_2) = k(y_1, y_2)$ and

$$\sigma \ : \ y_1 \longmapsto y_1, \quad y_2 \longmapsto \frac{y_1(y_1 + y_2)}{y_1 + y_2 + y_1y_2}.$$

Also define

$$z_1 = y_1, \quad z_2 = \frac{y_1 + y_2}{y_2}.$$

It follows that $k(y_1, y_2) = k(z_1, z_2)$ and

$$\sigma \ : \ z_1 \longmapsto z_1, \quad z_2 \longmapsto z_1z_2^{-1}.$$

Therefore we obtain

$$k(x_1, x_2)^{\langle\sigma\rangle} = k(z_1, z_2)^{\langle\sigma\rangle} = k\left(z_1, z_2 + \frac{z_1}{z_2}\right) = k\left(x_1 + x_2 + x_2^2, \frac{x_1^2 + x_1x_2^2 + x_2^3}{x_2(x_1 + x_2^2)}\right).$$

**Step 2.** In this step, we assume that $\operatorname{char} k = 3$. Note that the automorphism $\sigma$ becomes

$$x_1 \longmapsto \frac{x_1x_2^6}{(x_1^2 + x_1x_2^2 + x_2^3)^2}, \; x_2 \longmapsto \frac{-x_2^4}{x_1^2 + x_1x_2^2 + x_2^3}.$$

Define

$$y_1 = x_1x_2^{-2}, \quad y_2 = x_2^{-1}.$$

It follows that $k(x_1, x_2) = k(y_1, y_2)$ and

$$\sigma \ : \ y_1 \longmapsto y_1, \quad y_2 \longmapsto -y_2 - y_1 - y_1^2.$$

Hence we get

$$k(x_1,x_2)^{\langle\sigma\rangle} = k(y_1,y_2)^{\langle\sigma\rangle} = k(y_1, y_2(y_2+y_1+y_1^2)) = k\left(\frac{x_1}{x_2^2}, \frac{x_1^2+x_1x_2^2+x_2^3}{x_2^5}\right).$$

**4. The case char $k = 2$.** In this section, we assume that char $k=2$. Recall that the automorphism $\sigma$ is

$$x_1 \longmapsto \frac{x_1(x_1+x_2+x_2^2)^3}{(x_1^2+x_1x_2+x_2^3)^2}, \quad x_2 \longmapsto \frac{(x_1+x_2^2)(x_1+x_2+x_2^2)}{x_1^2+x_1x_2+x_2^3}.$$

Define

(9) $$y_1 = x_1, \quad y_2 = x_1+x_2+x_2^2, \quad y_3 = x_1+x_1x_2+x_2^3.$$

With the aid of computers, it is easy to see that

$$\sigma \ : \ y_1 \longmapsto y_1y_2^3y_3^{-2}, \ y_2 \longmapsto y_2, \ y_3 \longmapsto y_2^3y_3^{-1}.$$

From (9), we find that

(10) $$x_2 = \frac{y_2+y_3}{1+y_2}.$$

And therefore we have that $k(y_1,y_2,y_3) = k(x_1,x_2)$. Using (9) to eliminate $x_1, x_2$, we obtain the relation

(11) $$f(y_1,y_2,y_3) = y_1 + y_1y_2^2 + y_2^3 + y_3 + y_2y_3 + y_3^2 = 0.$$

Define

$$z_1 = y_1y_3^{-1}, \quad z_2 = y_2^2y_3^{-1}, \quad z_3 = y_2^{-1}y_3.$$

It follows that $k(y_1,y_2,y_3) = k(z_1,z_2,z_3)$ and

$$\sigma \ : \ z_1 \longmapsto z_1, \quad z_2 \longmapsto z_3 \longmapsto z_2.$$

We find that the relation $f(y_1,y_2,y_3) = 0$ in (11) becomes

(12) $$g(z_1,z_2,z_3) = 1 + z_1 + z_2z_3 + z_2^2z_3 + z_2z_3^2 + z_1z_2^2z_3^2 = 0.$$

Define

$$u_1 = z_1, \quad u_2 = z_2z_3, \quad u_3 = z_2+z_3.$$

Then we have $k(x_1,x_2)^{\langle\sigma\rangle} = k(z_1,z_2,z_3)^{\langle\sigma\rangle} = k(u_1, u_2, u_3)$ and the relation in (12) becomes

$$1 + u_1 + u_2 + u_1u_2^2 + u_2u_3 = 0.$$

Thus $u_3 \in k(u_1,u_2)$. It follows that $k(x_1,x_2)^{\langle\sigma\rangle} = k(u_1,u_2,u_3) = k(u_1,u_2)$ is rational over $k$. It is easy to obtain the formulae of the generators $u_1, u_2$ of $k(x_1,x_2)^{\langle\sigma\rangle}$ in terms of $x_1, x_2$. Indeed we have

$$u_1 = \frac{x_1}{x_1+x_1x_2+x_2^3}, \quad u_2 = x_1+x_2+x_2^2.$$

**5. The case char $k = 3$.** In this section, we assume that char $k=3$. Recall that the automorphism $\sigma$ is

$$x_1 \longmapsto \frac{x_1x_2^6}{(x_1^2+x_1x_2^2+x_2^3)^2}, \ x_2 \longmapsto \frac{-x_2^4}{x_1^2+x_1x_2^2+x_2^3}.$$

Define

$$y_1 = x_1, \quad y_2 = -x_2, \quad y_3 = x_1^2+x_1x_2^2+x_2^3.$$

It is clear that $k(x_1,x_2) = k(y_1,y_2,y_3)$ and

$$\sigma \ : \ y_1 \longmapsto y_1y_2^6y_3^{-2}, \ y_2 \longmapsto y_2^4y_3^{-1}, \ y_3 \longmapsto y_2^{15}y_3^{-4}.$$

The map of $\sigma$ above can be simplified as follows: Define

$$z_1 = y_1y_2^{-2}, \quad z_2 = y_2^{-4}y_3, \quad z_3 = y_2^{-1}.$$

It follows that $k(y_1,y_2,y_3) = k(z_1,z_2,z_3)$ and

$$\sigma \ : \ z_1 \longmapsto z_1, \quad z_2 \longmapsto z_3 \longmapsto z_2.$$

We also obtain the relation

(13) $$g(z_1,z_2,z_3) = z_1 + z_1^2 - z_2 - z_3 = 0.$$

Thus $k(x_1,x_2)^{\langle\sigma\rangle} = k(z_1,z_2,z_3)^{\langle\sigma\rangle} = k(u_1,u_2,u_3)$ where $u_1, u_2, u_3$ are defined by

$$u_1 = z_1, \quad u_2 = z_2z_3, \quad u_3 = z_2+z_3.$$

The relation $g(z_1,z_2,z_3) = 0$ in (13) becomes

$$u_1 + u_1^2 - u_3 = 0.$$

We conclude that $k(x_1,x_2)^{\langle\sigma\rangle} = k(u_1,u_2,u_3) = k(u_1,u_2)$ is rational over $k$. The generators $u_1, u_2$ of $k(x_1,x_2)^{\langle\sigma\rangle}$ over $k$ are given in terms of $x_1, x_2$ as follows:

$$u_1 = \frac{x_1}{x_2^2}, \quad u_2 = \frac{-(x_1^2+x_1x_2^2+x_2^3)}{x_2^5}.$$

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## References

[HK1] M. Hajja and M. Kang, Finite group actions on rational function fields, J. Algebra **149** (1992), no. 1, 139–154.

[HK2] M. Hajja and M. Kang, Three-dimensional purely monomial actions, J. Algebra **170** (1994), no. 3, 805–860.

[HM] A. Hoshi and K. Miyake, Tschirnhausen transformation of a cubic generic polynomial and a 2-dimensional involutive Cremona transformation, Proc. Japan Acad. Ser. A Math. Sci. **83** (2007), no. 3, 21–26.

[ HR ] A. Hoshi and Y. Rikuna, Rationality problem of three-dimensional purely monomial group actions: the last case, Math. Comp. **77** (2008), no. 263, 1823–1829.

[ KP ] I. Ya. Kolpakov-Miroshnichenko and Yu. G. Prokhorov, Rationality of fields of invariants of some four-dimensional linear groups, and an equivariant construction connected with the Segre cubic, Mat. Sb. **182** (1991), no. 10, 1430–1445; translation in Math. USSR-Sb. **74** (1993), no. 1, 169–183.

[ Sw ] R. Swan, Noether's problem in Galois theory, in *Emmy Noether in Bryn Mawr* (*Bryn Mawr, Pa.*, 1982), edited by B. Srinivasan and J. Sally, 21–40, Springer-Verlag, New York, 1983.

[ Wo ] S. Wolfram, *The Mathematica Book, Fifth Edition*, Wolfram Media, Inc., Cambridge University Press, 2003.

[ Za ] O. Zariski, On Castelnuovo's criterion of rationality $p_a = P_2 = 0$ of an algebraic surface, Illinois J. Math. **2** (1958), 303–315.