Yang-Mills connections with Weyl structure

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Abstract: In this paper, we treat with an arbitrary given connection D which is not necessarily *metric* or *torsion-free* in the tangent bundle TM over an n-dimensional closed (compact and connected) Riemannian manifold (M,g). We find the fact that if any connection D with Weyl structure (D,g,ω) relative to a 1-form ω in the tangent bundle is a Yang-Mills connection, then $d\omega = 0$. Moreover under the assumption $\sum_{i=1}^{n}[\alpha(e_i), R^D(e_i, X)] = 0$ $(X \in \mathfrak{X}(M))$, a necessary and sufficient condition for any connection D with Weyl structure (D,g,ω) to be a Yang-Mills connection is $\delta_\nabla R^D = 0$, where $\{e_i\}_{i=1}^n$ is an (locally defined) orthonormal frame on (M,g) and $D - \nabla = \alpha \in \Gamma(\bigwedge TM^* \otimes \mathrm{End}(TM))$, and ∇ is the Levi-Civita connection for g of (M,g).

Key words: Yang-Mills connection; conjugate connection; Weyl structure.

1. Introduction. In the theory of Yang-Mills connections, only the *metric* connections in a vector bundle have been treated so far. However, recently, Dragomir, Ichiyama and Urakawa (cf. [2]) developed a new Yang-Mills theory for *arbitrary* connections D in a vector bundle E with bundle metric h over a Riemannian manifold, not necessarily satisfying a metric connection, by using the concept of *conjugate connection* (cf. [6]). Precisely, if D is a connection in a vector bundle $E \longrightarrow M$, then the connection D^* given by

$$h(D^*_X s, t) = X(h(s,t)) - h(s, D_X t), \quad X \in \mathfrak{X}(M) \text{ and } s, t \in \Gamma(E), \tag{1.1}$$

is referred to as *conjugate* to D. Let (M,g) be a closed (compact and connected) Riemannian manifold. A Yang-Mills connection is a critical point of the Yang-Mills functional

$$\mathcal{YM}(D) = \frac{1}{2}\int_M \|R^D\|^2\, v_g \tag{1.2}$$

on the space $\mathfrak{C}_E$ of all connections in E, where R^D is the curvature tensor field for $D \in \mathfrak{C}_E$. Equivalently, D is a Yang-Mills connection if it satisfies the Yang-Mills equation (cf. [7,8,16])

$$\delta_D R^D = 0 \tag{1.3}$$

(the Euler-Lagrange equations of the variational principle associated with (1.2)). Note that, even if a connection D is torsion-free, then the conjugate connection D^* is not torsion-free in general. In fact, if (D,g) is a Weyl structure (cf. [4,5,10–15]) relative to a 1-form ω on (M,g) which is torsion-free, then D^* is not torsion-free in general (otherwise $\omega = 0$, hence $D = D^*$ is the Levi-Civita connection of (M,g)). From this point of view, D and D^* have different properties.

Recently, the present author obtained the following

Theorem A [9]. *A connection D in a vector bundle E over a closed Riemannian manifold (M,g) is a Yang-Mills connection if and only if the conjugate connection D^* is a Yang-Mills connection.*

In this paper, we treat with an arbitrary given connection D which is not necessarily *metric* or *torsion-free* in the tangent bundle TM over an n-dimensional closed Riemannian manifold (M,g). In §2, using properties of a connection D in the tangent bundle $E = TM$ over a closed Riemannian manifold (M,g) which has a Weyl structure (D,g,ω), i.e., $Dg = \omega \otimes g$, where ω is a 1-form on M, we get the following

Theorem 1. *Let (M,g) be a closed Riemannian manifold, and (D,g,ω) a Weyl structure in the tangent bundle TM over (M,g). Then,*

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$$(\delta_{D^*}R^{D^*} - \delta_D R^D)(X)Y = (\delta d\omega)(X)Y,$$
$$(X \in \mathfrak{X}(M), \quad Y \in \Gamma(E)).$$

By virtue of Theorem 1 and Theorem A, we obtain

Corollary 2. *If D is a Yang-Mills connection with Weyl structure (D, g, ω) in the tangent bundle TM over a closed Riemannian manifold (M, g), then $d\omega = 0$.*

In §3, we get the following

Theorem 3. *Let D be a connection with Weyl structure (D, g, ω) in the tangent bundle over a closed Riemannian manifold (M, g), and ∇ the Levi-Civita connection of (M, g). Assume $\sum_{i=1}^n [\alpha(e_i), R^D(e_i, X)] = 0$, where $X \in \mathfrak{X}(M)$ and $D - \nabla = \alpha$ and $\{e_i\}_{i=1}^n$ is an (locally defined) orthonormal frame on (M, g). Then, the following statements are equivalent*:
(*i*) *D is a Yang-Mills connection.*
(*ii*) $\delta_\nabla R^D = 0$.

2. The proof of Theorem 1 and Corollary 2. This section consists of two subsections. In the first subsection, we treat the Yang-Mills equation in vector bundles over a closed Riemannian manifold (M, g), using the concept of conjugate connection. And then, in the second subsection we prove Theorem 1 and Corollary 2.

2.1. Let E be a vector bundle, with bundle metric h, over an n-dimensional closed Riemannian manifold (M, g). Let $D \in \mathfrak{C}_E$ and ∇ the Levi-Civita connection of (M, g). The pair (D, ∇) induces a connection in product bundles $\bigwedge^p TM^* \otimes E$, also denoted by D. Set $A^p(E) := \Gamma(\bigwedge^p TM^* \otimes E)$. We consider the differential operator

$$d_D : A^p(E) \longrightarrow A^{p+1}(E),$$
$$(d_D\varphi)(X_1, X_2, \cdots, X_{p+1})$$
$$= \sum_{i=1}^{p+1} (-1)^{i+1} (D_{X_i}\varphi)(X_1, \cdots, \widehat{X}_i, \cdots, X_{p+1}),$$
$$\varphi \in A^p(E),\ X_i \in \mathfrak{X}(M)\ (i = 1, 2, \cdots, p+1),$$

which are defined by

$$d_D(\omega \otimes \xi) := d\omega \otimes \xi + (-1)^p \omega \wedge D\xi,$$
$$D_X(\omega \otimes \xi) := (\nabla_X \omega) \otimes \xi + \omega \otimes D_X \xi,$$

for $\omega \in \Gamma(\bigwedge^p TM^*)$, $\xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$.

Let δ_D be the formal adjoint of d_D with respect to the L^2-inner product

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle v_g$$

for φ, $\psi \in A^p(E)$. Here $\langle\ ,\ \rangle$ is the bundle metric in $\bigwedge^p TM^* \otimes E$ induced by the pair (g, h) and v_g is the canonical volume form on (M, g). The following identity is elementary, yet crucial (cf. [2])

$$\begin{aligned} \delta_D\varphi &= (-1)^{p+1}(*^{-1} \cdot d_{D^*} \cdot *)(\varphi) \\ &= (-1)^{np+1}(* \cdot d_{D^*} \cdot *)(\varphi) \end{aligned} \tag{2.1}$$

for any $\varphi \in A^{p+1}(E)$. Here, $* : A^q(E) \longrightarrow A^{n-q}(E)$, $(0 \leq q \leq n)$, is the Hodge operator with respect to g. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame on (M, g) and $\{\theta^j\}_{j=1}^n$ the dual coframe. Let $\{d_\alpha\}_{\alpha=1}^\gamma$ be a local orthonormal frame of (E, h) and $\{\sigma^\alpha\}_{\alpha=1}^\gamma$ the dual coframe, where γ is the rank of E. Using the orthonormal frame and the dual frame on (M, g), and the orthonormal frame and the dual frame of (E, h), and properties of a connection in a smooth vector bundle E over (M, g), we proved Theorem A (cf. [9]). Note that (2.1) may also be written as (cf. [2,3])

$$\begin{aligned} &(\delta_D\varphi)(X_1, \cdots, X_p) \\ &\qquad = -\sum_{i=1}^n (D^*_{e_i}\varphi)(e_i, X_1, \cdots, X_p). \end{aligned} \tag{2.2}$$

The connections D, $D^* \in \mathfrak{C}_E$ naturally induce connections, denoted by the same symbols, in $\mathrm{End}(E)$ $(:= E \otimes E^*)$. Then, a straightforward argument shows that D, $D^* \in \mathfrak{C}_{\mathrm{End}(E)}$ are conjugate connections. The following curvature property is immediate (cf. [1, Proposition 2.1])

$$h(R^D(X, Y)s, t) = -h(s, R^{D^*}(X, Y)t), \tag{2.3}$$

for $s, t \in \Gamma(E)$ and $X, Y \in \mathfrak{X}(M)$.

Now, we find from (1.3) and (2.2) that the connection D^* is a Yang-Mills connection if and only if

$$-\sum_i (D_{e_i} R^{D^*})((e_i, \cdot), \cdot) = 0. \tag{2.4}$$

2.2. Let D be a connection with Weyl structure which is not necessarily *metric* or *torsion-free* in the tangent bundle TM over an n-dimensional closed Riemannian manifold (M, g), that is,

$$Dg = \omega \otimes g. \tag{2.5}$$

In this case, we have the following properties:

$$D^*{}_X Y = D_X Y + \omega(X)Y$$

(2.6) $$(X \in \mathfrak{X}(M),\ Y \in \Gamma(TM)),$$

(2.7) $$\begin{cases} R^{D^*}(X,Y)Z = R^D(X,Y)Z + d\omega(X,Y)Z \\ (X,Y \in \mathfrak{X}(M),\ Z \in \Gamma(TM)), \\ D^*g = -Dg. \end{cases}$$

By virtue of (2.2) and (2.7), for $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(TM)$

$$\begin{aligned}
&(\delta_{D^*} R^{D^*})(X)Y \\
&= -\sum_{i=1}^n (D_{e_i} R^{D^*})(e_i, X)Y \\
&= -\sum_{i=1}^n \{D_{e_i}(R^{D^*}(e_i,X)Y) - R^{D^*}(\nabla_{e_i} e_i, X)Y \\
&\quad - R^{D^*}(e_i, \nabla_{e_i} X)Y - R^{D^*}(e_i,X)D_{e_i}Y\} \\
&= -\sum_{i=1}^n \{D_{e_i}(R^D(e_i,X)Y + d\omega(e_i,X)Y) \\
&\quad - R^D(\nabla_{e_i} e_i, X)Y - d\omega(\nabla_{e_i} e_i, X)Y \\
&\quad - R^D(e_i, \nabla_{e_i} X)Y - d\omega(e_i, \nabla_{e_i} X)Y \\
&\quad - R^D(e_i,X)D_{e_i}Y - d\omega(e_i,X)D_{e_i}Y\}.
\end{aligned}$$

From (2.2) and (2.6), we get for $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(TM)$

$$\begin{aligned}
&(\delta_D R^D)(X)Y \\
&= -\sum_{i=1}^n (D^*_{e_i} R^D)(e_i, X)Y \\
&= -\sum_{i=1}^n \{D^*_{e_i}(R^D(e_i,X)Y) - R^D(\nabla_{e_i} e_i, X)Y \\
&\quad - R^D(e_i, \nabla_{e_i} X)Y - R^D(e_i,X)D^*_{e_i}Y\} \\
&= -\sum_{i=1}^n \{D_{e_i}(R^D(e_i,X)Y) - R^D(\nabla_{e_i} e_i, X)Y \\
&\quad - R^D(e_i, \nabla_{e_i} X)Y - R^D(e_i,X)D_{e_i}Y\}.
\end{aligned}$$

Consequently, for $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(TM)$ we have

$$\begin{aligned}
&(\delta_{D^*} R^{D^*})(X)Y - (\delta_D R^D)(X)Y \\
&= -\sum_{i=1}^n \{D_{e_i}(d\omega(e_i,X)Y) - d\omega(\nabla_{e_i} e_i, X)Y \\
&\quad - d\omega(e_i, \nabla_{e_i} X)Y - d\omega(e_i,X)D_{e_i}Y\} \\
&= -\sum_{i=1}^n (D_{e_i} d\omega)(e_i,X)Y = -\sum_{i=1}^n (\nabla_{e_i} d\omega)(e_i,X)Y \\
&= (\delta_\nabla d\omega)(X)Y = (\delta d\omega)(X)Y.
\end{aligned}$$

Thus, the proof of Theorem 1 is completed. Since X is arbitrary in $\mathfrak{X}(M)$ and Y is arbitrary in $\Gamma(TM)$, by virtue of Theorem 1 and Theorem A, Corollary 2 is obtained.

3. The proof of Theorem 3. Let (D,g) be a Weyl structure with respect to the 1-form ω on M. We put $D_X Y - \nabla_X Y =: \alpha(X)Y$, $(X \in \mathfrak{X}(M)$ and $Y \in \Gamma(TM))$. From (2.6) and the definition of $\alpha \in \Gamma(\bigwedge TM^* \otimes \mathrm{End}TM)$, we have

(3.1) $$\begin{aligned} &D^*{}_X Y = \nabla_X Y + \alpha(X)Y + \omega(X)Y, \\ &(X \in \mathfrak{X}(M) \quad \text{and} \quad Y \in \Gamma(TM)). \end{aligned}$$

Moreover, we get the following

Lemma 3.1. *$g(\alpha(X)Y,Z) = g(Y, -\alpha(X)Z - \omega(X)Z)$, $(X \in \mathfrak{X}(M); Y,Z \in \Gamma(TM))$.*

Proof. By virtue of the fact $Dg = \omega \otimes g$,

$$\begin{aligned}
&g(\alpha(X)Y,Z) \\
&= g(D_X Y - \nabla_X Y, Z) \\
&= X(g(Y,Z)) - (D_X g)(Y,Z) \\
&\quad - g(Y, D_X Z) - X(g(Y,Z)) + g(Y, \nabla_X Z) \\
&= g(Y, -\alpha(X)Z - \omega(X)Z).
\end{aligned}$$

Thus, the proof of this Lemma is completed.

From the fact $D^*g = -Dg = -\omega \otimes g$, we get for $X \in \mathfrak{X}(M)$ and $Y,Z \in \Gamma(TM)$

$$\begin{aligned}
&g((\delta_D R^D)(X)Y, Z) \\
&= -\sum_{i=1}^n g((D^*_{e_i} R^D)(e_i,X)Y, Z) \\
&= -\sum_{i=1}^n \{e_i(g(R^D(e_i,X)Y,Z)) \\
&\quad + \omega(e_i) g(R^D(e_i,X)Y,Z) \\
&\quad - g(R^D(\nabla_{e_i} e_i, X)Y, Z) - g(R^D(e_i, \nabla_{e_i} X)Y, Z) \\
&\quad - g(R^D(e_i,X)D^*_{e_i}Y, Z) - g(R^D(e_i,X)Y, D^*_{e_i}Z)\}.
\end{aligned}$$

By virtue of (3.1), the equation above changes as follows:

(3.2) $$\begin{aligned}
&g((\delta_D R^D)(X)Y, Z) \\
&= -\sum_{i=1}^n \{g((\nabla_{e_i} R^D)(e_i,X)Y, Z) \\
&\quad - g(R^D(e_i,X)\ \alpha(e_i)Y, Z) \\
&\quad - g(R^D(e_i,X)Y, \alpha(e_i)Z) \\
&\quad - \omega(e_i)\ g(R^D(e_i,X)Y, Z)\}.
\end{aligned}$$

Consequently, from (2.2), (3.2) and Lemma 3.1, we obtain

$$g((\delta_D R^D)(X)Y, Z)$$
$$= g((\delta_\nabla R^D)(X)Y - \sum_{i=1}^{n}[\alpha(e_i), R^D(e_i, X)]Y, Z),$$

where $X \in \mathfrak{X}(M)$ and $Y, Z \in \Gamma(TM)$. Since Y is arbitrary in $\Gamma(TM)$ and X is arbitrary in $\mathfrak{X}(M)$, the proof of Theorem 3 is completed.

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