Disconnected Julia sets of quartic polynomials and a new topology of the symbol space

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Abstract: For a certain quartic polynomial, there exists a homeomorphism between the set of all components of the filled-in Julia set with the Hausdorff metric and some subset of the corresponding symbol space with the ordinary metric. But these sets are not compact with respect to each metric. We introduce a new topology with respect to which these sets are compact.

Key words: Quartic polynomials; disconnected Julia sets; symbolic dynamics.

1. Introduction and the main results.

Let $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ be the Riemann sphere and let $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be a rational function of degree $d \geq 2$. In the theory of complex dynamics, there are two important sets called the Fatou set $F(f)$ and the Julia set $J(f)$. The Fatou set $F(f)$ is the set of normality in the sense of Montel for the family $\{f^n\}_{n=0}^{\infty}$, where $f^n = f \circ \cdots \circ f$. The Julia set $J(f)$ is the complement $\hat{\mathbf{C}} \setminus F(f)$. $J(f)$ is either connected or else has uncountably many connected components. In the case that f is a polynomial, we define the filled-in Julia set $K(f)$ as

$$K(f) = \{z \in \mathbf{C} : \{f^n(z)\}_{n=0}^{\infty} \text{ is bounded}\}.$$

$J(f)$ is the topological boundary of $K(f)$. We call $A(f) = \hat{\mathbf{C}} \setminus K(f)$ the attracting basin of the point at infinity.

We often consider some model in order to simplify the dynamics of f. The model is the symbol space and the shift map which defines a dynamical system on the symbol space. Let X^{ω} be the countable product of a set X

Definition 1.1. The *symbol space* on q symbols is the countable product $\Sigma_q = \{1, 2, \ldots, q\}^{\omega}$. For $s = (s_n)$ and $t = (t_n) \in \Sigma_q$, the metric ρ on Σ_q is defined as

$$\rho(s,t) = \sum_{n=0}^{\infty} \frac{\delta(s_n, t_n)}{2^n}, \quad \text{where } \delta(k,l) = \begin{cases} 1 & \text{if } k \neq l, \\ 0 & \text{if } k = l. \end{cases}$$

Then (Σ_q, ρ) is a compact metric space. The *shift map* $\sigma : \Sigma_q \to \Sigma_q$ is defined as

$$\sigma((s_0, s_1, s_2, \ldots)) = (s_1, s_2, \ldots).$$

The shift map σ is continuous with respect to the metric ρ.

The connectivity of the Julia set $J(f)$ is affected by the behavior of finite critical points.

Theorem 1.2 [4, pp.11–12]**.** *Let f be a polynomial of degree $d \geq 2$. If all finite critical points of f are in $A(f)$, then $J(f)$ is totally disconnected. Furthermore $f|_{J(f)}$ is topologically conjugate to the shift map $\sigma : \Sigma_d \to \Sigma_d$. On the other hand, $J(f)$ is connected if and only if all finite critical points of f are in $K(f)$.*

If some critical orbits of a polynomial converge to the point at infinity but not all critical orbits converge to it, then the Julia set is disconnected and not generally totally disconnected. We can simplify the dynamics of some quartic polynomial on the Julia set when the Julia set is disconnected and not totally disconnected, see [1].

Definition 1.3. Let f be a polynomial of degree $d \geq 2$. The *Green's function* associated with $K(f)$ is defined as

$$G(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n(z)|,$$

where $\log^+ x = \max\{\log x, 0\}$. $G(z)$ is zero for $z \in K(f)$ and positive for $z \in \mathbf{C} \setminus K(f)$. Note that G satisfies the identity $G(f(z)) = d{\cdot}G(z)$.

Definition 1.4. The triple (f, U, V) consist-

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ing of bounded simply connected domains U and V such that $\overline{U} \subset V$ and a holomorphic proper map $f : U \to V$ of degree d, is called *polynomial-like map* of degree d. The *filled-in Julia set* $K(f)$ of a polynomial-like map (f, U, V) is defined as

$$K(f) = \{z \in U : \{f^n(z)\}_{n=0}^{\infty} \subset U\}.$$

Let f be a quartic polynomial and let c_1, c_2 and c_3 be finite critical points of f. Let G be the Green's function associated with the filled-in Julia set $K(f)$. Suppose that $G(c_1) = G(c_2) = 0$ and $G(c_3) > 0$, that is, $c_1, c_2 \in K(f)$ and $c_3 \in A(f)$.

Let U be the bounded component of $\mathbf{C} \setminus G^{-1}(G(f(c_3)))$. Suppose that U_A and U_B are the different bounded components of $\mathbf{C} \setminus G^{-1}(G(c_3))$ such that $c_1 \in U_A$ and $c_2 \in U_B$. Then U_A and U_B are proper subsets of U. Furthermore $(f|_{U_A}, U_A, U)$ and $(f|_{U_B}, U_B, U)$ are polynomial-like maps of degree 2. Suppose that filled-in Julia sets $K_A = K(f|_{U_A})$ and $K_B = K(f|_{U_B})$ are connected.

Let $K(f)^*$ be the set of all components of $K(f)$. Since c_3 is in $A(f)$, the Julia set $J(f)$ and the filled-in Julia set $K(f)$ are disconnected and have uncountably many components respectively. Therefore $K(f)^*$ is an uncountable set. $K(f)^*$ becomes a metric space with the Hausdorff metric d_H. We define a map $F : (K(f)^*, d_H) \to (K(f)^*, d_H)$ as $F(K) = f(K)$ for $K \in K(f)^*$. Then F is continuous.

Let $\Sigma_6 = \{1, 2, 3, 4, \mathsf{A}, \mathsf{B}\}^{\omega}$ be the symbol space which we treat mainly in this paper. We define a subset Σ of Σ_6 as follows: $s = (s_n) \in \Sigma$ if and only if

(S1) if $s_n = \mathsf{A}$, then $s_{n+1} = \mathsf{A}$,
(S2) if $s_n = \mathsf{B}$, then $s_{n+1} = \mathsf{B}$,
(S3) if $s_n = \mathsf{A}$ and $s_{n-1} \neq \mathsf{A}$, then $s_{n-1} = 3$ or 4,
(S4) if $s_n = \mathsf{B}$ and $s_{n-1} \neq \mathsf{B}$, then $s_{n-1} = 1$ or 2,
(S5) if $s \in \Sigma_4 = \{1, 2, 3, 4\}^{\omega}$, then there exist subsequences $(s_{n(k)})_{k=1}^{\infty}$ and $(s'_{n(l)})_{l=1}^{\infty}$ such that $s_{n(k)} = 1$ or 2 for all $k \geq 1$ and $s'_{n(l)} = 3$ or 4 for all $l \geq 1$.

The author proved the following theorems in [1].

Theorem 1.5. *Assume that filled-in Julia sets K_A and K_B are connected. Then there exists a homeomorphism $\Lambda : (K(f)^*, d_H) \to (\Sigma, \rho)$ such that $\Lambda \circ F = \sigma \circ \Lambda$.*

Theorem 1.6. *Under the assumption of Theorem 1.5, there exist polynomials g_1 and g_2 of degree 2 and a homeomorphism h on $K(f)$ such that*

$$h(J(f)) = J(\langle g_1, g_2 \rangle),$$

where $\langle g_1, g_2 \rangle$ is the polynomial semigroup generated by g_1 and g_2 and $J(\langle g_1, g_2 \rangle)$ is the Julia set of $\langle g_1, g_2 \rangle$.

Theorem 1.5 means that componentwise dynamics of f on $K(f)$ (of course, also on $J(f)$) can be simplified as dynamics of the shift map on Σ. The space (Σ, ρ) is not compact as the following example shows. The sequence

$$\left\{ s^{(n)} = (\underbrace{1, 1, \ldots, 1}_{n \text{ times}}, \mathsf{B}, \mathsf{B}, \mathsf{B}, \ldots) \right\}_{n=0}^{\infty}$$

in Σ converges to $s = (1, 1, 1, \ldots)$ but s is not in Σ. Since (Σ, ρ) is not compact, although the dynamical system $(K(f)^*, f)$ is conjugated by (Σ, σ) by Theorem 1.5, many good properties of the symbolic dynamical system are not available. So we impose a question: Is it possible to introduce a new topology on Σ which makes Σ compact and reflects the dynamical system $(K(f)^*, f)$ in a natural way? In this paper, we answer this question.

Theorem 1.7. *Let Σ be as above. Then there exists a topology $\mathcal{O}$ of Σ such that $(\Sigma, \mathcal{O})$ is compact, metrizable, perfect and totally disconnected. Moreover the shift map $\sigma : (\Sigma, \mathcal{O}) \to (\Sigma, \mathcal{O})$ is continuous.*

Regarding Λ in Theorem 1.5 just as a bijection between the sets $K(f)^*$ and Σ, we define $\mathcal{G}$ to be the quotient topology of $K(f)^*$ relative to Λ^{-1} and the topology $\mathcal{O}$ of Σ as in Theorem 1.7. Then $\Lambda : (K(f)^*, \mathcal{G}) \to (\Sigma, \mathcal{O})$ is a homeomorphism such that $\Lambda \circ F = \sigma \circ \Lambda$.

Corollary 1.8. *$(K(f)^*, \mathcal{G})$ is compact, metrizable, perfect and totally disconnected. Moreover $F : (K(f)^*, \mathcal{G}) \to (K(f)^*, \mathcal{G})$ is continuous.*

2. Known results in general topology.

For the following definitions and theorems we refer to [2,5].

Definition 2.1. Let X be a topological space. X is *sequentially compact* if every sequence of points of X contains a convergent subsequence. X is *countably compact* if every countable open covering of X has a finite subcovering. X is a *Lindelöf space* if every open covering of X has a countable subcovering.

Theorem 2.2. *If a topological space X is sequentially compact, then X is countably compact.*

Theorem 2.3. *If a topological space X satisfies the second axiom of countability, then X is a Lindelöf space.*

Theorem 2.4. *A topological space X is* *compact if and only if X is a countably compact Lindelöf space.*

Definition 2.5. Let X be a topological space. X is a T_1-*space* if for any distinct points x and $y \in X$, there exists an open neighborhood U of x such that $y \notin U$. X is a T_2-*space* or a *Hausdorff space* if for any distinct points x and $y \in X$, there exist open neighborhoods U of x and V of y such that $U \cap V = \emptyset$.

Definition 2.6. A T_1-space X is a *regular space* if for any $x \in X$ and any closed set L with $x \notin L$, there exists open neighborhoods U of x and V of L such that $U \cap V = \emptyset$.

Definition 2.7. A T_1-space X is a *normal space* if for any closed sets A and B of X with $A \cap B = \emptyset$, there exists open neighborhoods U of A and V of B such that $U \cap V = \emptyset$.

Theorem 2.8. *Each compact Hausdorff space is normal.*

Theorem 2.9. *A topological space X satisfies the second axiom of countability is metrizable if and only if X is a regular space.*

Definition 2.10. A topological space X is a 0-*dimensional space* if there exists an open basis $\mathcal{B}$ of X such that every $B \in \mathcal{B}$ is open and closed.

Theorem 2.11. *Let X be a compact Hausdorff space. X is a 0-dimensional space if and only if X is totally disconnected.*

3. Definition of a new topology of Σ. Our goal in this section is to prove Theorem 1.7. We define a topology of Σ. If $s = (\mathsf{A}, \mathsf{A}, \mathsf{A}, \ldots) \in \Sigma$, we define subsets $N_s^{(k)}$ of Σ as

$$N_s^{(k)} = \{s\} \cup \{t = (t_n) \in \Sigma : t_n = 1 \text{ or } 2 \text{ for } n \leq k\}.$$

Similarly, if $s = (\mathsf{B}, \mathsf{B}, \mathsf{B}, \ldots) \in \Sigma$,

$$N_s^{(k)} = \{s\} \cup \{t = (t_n) \in \Sigma : t_n = 3 \text{ or } 4 \text{ for } n \leq k\}.$$

If $s = (s_0, \ldots, s_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \ldots) \in \Sigma$ with $s_l \neq \mathsf{A}$, $N_s^{(k)}$ is the union of $\{s\}$ and

$$\left\{(t_n) \in \Sigma : t_n = \begin{cases} s_n & \text{if } n \leq l, \\ 1 \text{ or } 2 & \text{if } l+1 \leq n \end{cases} \text{ for } n \leq k\right\}.$$

Similarly, if $s = (s_0, \ldots, s_l, \mathsf{B}, \mathsf{B}, \mathsf{B}, \ldots) \in \Sigma$ with $s_l \neq \mathsf{B}$, $N_s^{(k)}$ is the union of $\{s\}$ and

$$\left\{(t_n) \in \Sigma : t_n = \begin{cases} s_n & \text{if } n \leq l, \\ 3 \text{ or } 4 & \text{if } l+1 \leq n \end{cases} \text{ for } n \leq k\right\}.$$

Finally, if $s = (s_n) \in \Sigma \cap \Sigma_4$,

$$N_s^{(k)} = \{t = (t_n) \in \Sigma : t_n = s_n \text{ for } n \leq k\}.$$

Note that $N_s^{(k+1)} \subset N_s^{(k)}$ for all $s \in \Sigma$ and $k \geq 0$. Let $\mathcal{N}(s) = \{N_s^{(k)}\}_{k=0}^{\infty}$ and $\mathcal{N} = \{\mathcal{N}(s) : s \in \Sigma\}$.

Lemma 3.1. *$\mathcal{N}$ is a neighborhood system of Σ.*

Proof. Let $s \in \Sigma$. (i) If $N \in \mathcal{N}(s)$, then $s \in N$. (ii) For N_1 and $N_2 \in \mathcal{N}(s)$, there exist k_1 and $k_2 \geq 0$ such that $N_1 = N_s^{(k_1)}$ and $N_2 = N_s^{(k_2)}$. Fix $k \geq \max\{k_1, k_2\}$ and let $N_3 = N_s^{(k)}$. Then $N_3 \in \mathcal{N}(s)$ and $N_3 \subset N_1 \cap N_2$. (iii) For $N \in \mathcal{N}(s)$, there exists $k \geq 0$ such that $N = N_s^{(k)}$. For $t \in N$, let $N' = N_t^{(k)}$. Then $N' \in \mathcal{N}(t)$ and $N' \subset N$. $\square$

Therefore $(\Sigma, \mathcal{O})$ is a topological space, where $\mathcal{O}$ is the topology generated by $\mathcal{N}$. We obtain immediately the following lemmas.

Lemma 3.2. *$(\Sigma, \mathcal{O})$ satisfies the first axiom of countability.*

Proof. We choose a neighborhood basis of $s \in \Sigma$ as $\mathcal{N}(s)$. $\square$

Lemma 3.3. *$(\Sigma, \mathcal{O})$ is a Hausdorff space.*

Proof. For distinct points $s = (s_n)$ and $t = (t_n) \in \Sigma$, there exist $k \geq 0$ such that $s_k \neq t_k$. Let $M = N_s^{(k)} \in \mathcal{N}(s)$ and $N = N_t^{(k)} \in \mathcal{N}(t)$. Then $M \cap N = \emptyset$. $\square$

Lemma 3.4. *$(\Sigma, \mathcal{O})$ is perfect.*

Proof. For $s \in \Sigma$ and any neighborhood $O \in \mathcal{O}$ of s, there exists $N \in \mathcal{N}(s)$ such that $N \subset O$. It is clear that $(O \setminus \{s\}) \cap \Sigma \supset (N \setminus \{s\}) \cap \Sigma \neq \emptyset$. $\square$

We show that $(\Sigma, \mathcal{O})$ is compact from now on. By Theorem 2.2, 2.3 and 2.4, we need only to show that $(\Sigma, \mathcal{O})$ is sequentially compact and satisfies the second axiom of countability.

Lemma 3.5. *$(\Sigma, \mathcal{O})$ is sequentially compact.*

Proof. Let $\{s^{(k)} = (s_n^{(k)})\}_{k=1}^{\infty} \subset \Sigma$. We choose a subsequence $\{s^{\langle\alpha\rangle}\}_{\alpha=0}^{\infty}$ as follows: [Step 0] There exists a subsequence $\{s^{(k_l)}\}_{l=1}^{\infty}$ such that $s_0^{(k_l)} = s_0$ for $l \geq 1$, where $s_0 = 1$, 2, 3 or 4. Let $s^{\langle 0\rangle}$ be one of $s^{(k_l)}$. Then $s^{\langle 0\rangle} = (s_0, s_1^{\langle 0\rangle}, s_2^{\langle 0\rangle}, \ldots)$. We rewrite $s^{(k_l)}$ as $s^{(k)}$. [Step 1] There exists a subsequence $\{s^{(k_l)}\}_{l=1}^{\infty}$ such that $s_1^{(k_l)} = s_1$ for $l \geq 1$, where $s_1 = 1$, 2, 3 or 4. Let $s^{\langle 1\rangle}$ be one of $s^{(k_l)}$. Then $s^{\langle 1\rangle} = (s_0, s_1, s_2^{\langle 1\rangle}, s_3^{\langle 1\rangle}, \ldots)$. We rewrite $s^{(k_l)}$ as $s^{(k)}$. [Step α] Inductively, we can choose $s^{\langle\alpha\rangle} = (s_0, \ldots, s_\alpha, s_{\alpha+1}^{\langle\alpha\rangle}, s_{\alpha+2}^{\langle\alpha\rangle}, \ldots)$. Let $s = (s_0, s_1, s_2 \ldots)$. If $s \in \Sigma$, for a neighborhood $O \in \mathcal{O}$ of s, there exists $N = N_s^{(\alpha_0)} \in \mathcal{N}(s)$ such that $N \subset O$. If $\alpha \geq \alpha_0$, then $s^{\langle\alpha\rangle} \in N \subset O$. Therefore $s^{\langle\alpha\rangle}$ converges to s with respect to $\mathcal{O}$. If $s \notin \Sigma$, there exists a unique $\beta \geq 0$ such that (i) $s_{\beta-1} = 3$ or 4 and $s_n = 1$ or 2 for $\beta \leq n$ or (ii) $s_{\beta-1} = 1$ or 2 and $s_n = 3$ or 4 for $\beta \leq n$. If (i) is the case, let

$$t = \begin{cases} (\mathsf{A}, \mathsf{A}, \mathsf{A}, \ldots) & \text{if } \beta = 0, \\ (s_0, \ldots, s_{\beta-1}, \mathsf{A}, \mathsf{A}, \mathsf{A}, \ldots) & \text{if } \beta \geq 1. \end{cases}$$

If (ii) is the case, let

$$t = \begin{cases} (\mathsf{B}, \mathsf{B}, \mathsf{B}, \ldots) & \text{if } \beta = 0, \\ (s_0, \ldots, s_{\beta-1}, \mathsf{B}, \mathsf{B}, \mathsf{B}, \ldots) & \text{if } \beta \geq 1. \end{cases}$$

Then $s^{\langle\alpha\rangle}$ converges to t with respect to $\mathcal{O}$ by the same argument. Therefore $(\Sigma, \mathcal{O})$ is sequentially compact. □

Lemma 3.6. *$(\Sigma, \mathcal{O})$ satisfies the second axiom of countability.*

Proof. Let

$$\mathcal{B} = \bigcup_{s \in \Sigma \setminus \Sigma_4} \mathcal{N}(s).$$

First we show that $\mathcal{B}$ is an open basis of $(\Sigma, \mathcal{O})$. It is clear that $\mathcal{B} \subset \mathcal{O}$. Let $O \in \mathcal{O}$ and $s = (s_n) \in O$. If $s \in \Sigma \setminus \Sigma_4$, then there exists $M \in \mathcal{N}(s) \subset \mathcal{B}$ such that $s \in M \subset O$ by the definition of $\mathcal{O}$. If $s \in \Sigma \cap \Sigma_4$, there exists $N = N_s^{(k)} \in \mathcal{N}(s)$ such that $s \in N \subset O$ by the definition of $\mathcal{O}$. However we do not know yet whether $N \in \mathcal{B}$ at this stage. Let $t = (s_0, \ldots, s_k, t_{k+1}, t_{k+2}, \ldots) \in N \cap (\Sigma \setminus \Sigma_4)$ and $M = N_t^{(k)}$. Then $M \in \mathcal{B}$ and, in fact, $M = N$. Therefore $s \in M = N \subset O$. Consequently $\mathcal{B}$ is an open basis of $(\Sigma, \mathcal{O})$. The countability of $\mathcal{B}$ follows from that of $\Sigma \setminus \Sigma_4$ and $\mathcal{N}(s)$. □

We obtain the following lemma by Lemma 3.5 and 3.6.

Lemma 3.7. *$(\Sigma, \mathcal{O})$ is compact.*

By Theorem 2.8, $(\Sigma, \mathcal{O})$ is normal, especially $(\Sigma, \mathcal{O})$ is regular. Therefore we obtain the following lemma by Theorem 2.9.

Lemma 3.8. *$(\Sigma, \mathcal{O})$ is metrizable.*

Next, we show that $(\Sigma, \mathcal{O})$ is totally disconnected. By Theorem 2.11, we need only to show that $(\Sigma, \mathcal{O})$ is a 0-dimensional space.

Lemma 3.9. *Let $s \in \Sigma \setminus \Sigma_4$.*

(i) *If $s = (\mathsf{A}, \mathsf{A}, \mathsf{A}, \ldots)$ or $s = (\mathsf{B}, \mathsf{B}, \mathsf{B}, \ldots)$, then $N_s^{(k)}$ is open and closed for $k \geq 0$.*

(ii) *If $s = (s_0, \ldots, s_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \ldots)$ with $s_l \neq \mathsf{A}$ or $s = (s_0, \ldots, s_l, \mathsf{B}, \mathsf{B}, \mathsf{B}, \ldots)$ with $s_l \neq \mathsf{B}$, then $N_s^{(k)}$ is open and closed for $k \geq l+1$.*

Proof. (i) Let $s = (\mathsf{A}, \mathsf{A}, \mathsf{A}, \ldots)$ (resp. $s = (\mathsf{B}, \mathsf{B}, \mathsf{B}, \ldots)$). We show that $N_s^{(k)}$ is closed. Let $M = \Sigma \setminus N_s^{(k)}$ For $t = (t_n) \in M$, there exists $\alpha \leq k$ such that $t_\alpha \neq 1$ or 2 (resp. $t_\alpha \neq 3$ or 4). Then $N_s^{(k)} \cap N_t^{(\alpha)} = \emptyset$ and $N_t^{(\alpha)} \subset M$. Therefore M is open and $N_s^{(k)}$ is closed. (ii) The proof is similar to that of (i). □

For $s = (\mathsf{A}, \mathsf{A}, \mathsf{A} \ldots)$ or $s = (\mathsf{B}, \mathsf{B}, \mathsf{B} \ldots)$, let $\mathcal{N}'(s) = \mathcal{N}(s)$. And for $s = (s_0, \ldots, s_l, \mathsf{A}, \mathsf{A}, \ldots)$ with $s_l \neq \mathsf{A}$ or $s = (s_0, \ldots, s_l, \mathsf{B}, \mathsf{B}, \ldots)$ with $s_l \neq \mathsf{B}$, let $\mathcal{N}'(s) = \{N_s^{(k)} : k \geq l+1\}$. Let

$$\mathcal{B}' = \bigcup_{s \in \Sigma \setminus \Sigma_4} \mathcal{N}'(s).$$

Lemma 3.10. *$\mathcal{B}'$ is an open basis of $(\Sigma, \mathcal{O})$.*

Proof. The proof is smiler to that of Lemma 3.6. □

From Lemma 3.9 and Lemma 3.10, $(\Sigma, \mathcal{O})$ is a 0-dimensional space. Therefore we obtain the following lemma by Theorem 2.11.

Lemma 3.11. *$(\Sigma, \mathcal{O})$ is totally disconnected.*

Finally, we show the following lemma.

Lemma 3.12. *The shift map $\sigma : (\Sigma, \mathcal{O}) \to (\Sigma, \mathcal{O})$ is continuous.*

Proof. Let $s = (s_0, s_1, s_2, \ldots) \in \Sigma$. For a neighborhood $O \in \mathcal{O}$ of $\sigma(s) = (s_1, s_2, \ldots)$, there exists $N = N_{\sigma(s)}^{(k)} \in \mathcal{N}(\sigma(s))$ such that $N \subset O$. We take a neighborhood $M = N_s^{(k+1)}$ of s. Then $\sigma(M) = N \subset O$. Therefore $\sigma : (\Sigma, \mathcal{O}) \to (\Sigma, \mathcal{O})$ is continuous. □

We have completed the proof of Theorem 1.7.

4. Appropriateness of the convergence with respect to $\mathcal{O}$. We shall formulate $\Lambda : K(f)^* \to \Sigma$. We refer to [1] for the detailed proof. Let U, U_A, U_B, K_A and K_B be the same as the section 1. There exist forward invariant rays R_{A1} and R_{B1} under f such that R_{A1} lands at a point on ∂K_A and R_{B1} lands at a point on ∂K_B. These landing points are repelling or parabolic fixed points of f. Let R_{A2} and R_{B2} be components of $f^{-1}(R_{A1})$ and $f^{-1}(R_{B1})$ which satisfy $R_{A2} \cap U_A \neq \emptyset$ and $R_{B2} \cap U_B \neq \emptyset$ and differ from R_{A1} and R_{B1}. We set $V_A = U \setminus (K_A \cup R_{A1})$ and $V_B = U \setminus (K_B \cup R_{B1})$. Let I_1, I_2, I_3 and I_4 be branches of f^{-1} such that

$$I_1 : V_A \to U_1, \quad I_2 : V_A \to U_2,$$
$$I_3 : V_B \to U_3, \quad I_4 : V_B \to U_4,$$

where U_1 and U_2 are components of $U_A \setminus K_A \cup R_{A1} \cup R_{A2}$. Similarly, U_3 and U_4 are components of $U_B \setminus K_B \cup R_{B1} \cup R_{B2}$. We define $\Lambda : K(f)^* \to \Sigma$ as follows: for $K \in K(f)^*$,

$$[\Lambda(K)]_n = \begin{cases} i & \text{if } f^n(K) \subset U_i, \\ \mathsf{A} & \text{if } f^n(K) = K_A, \\ \mathsf{B} & \text{if } f^n(K) = K_B, \end{cases}$$

where $n \geq 0$ and $i = 1, 2, 3, 4$. We can also formulate $\Lambda^{-1} : \Sigma \to K(f)^*$ as follows: if $s_n = \mathsf{A}$ and $s_{n-1} \neq \mathsf{A}$,

$$\Lambda^{-1}(s) = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(K_A).$$

If $s_n = \mathsf{B}$ and $s_{n-1} \neq \mathsf{B}$,

$$\Lambda^{-1}(s) = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(K_B).$$

If $s \in \Sigma_4$, there exists a subsequence $(s_{n(l)})_{l=1}^{\infty}$ such that $s_{n(l)} = 1$ or 2 and $s_{n(l)-1} = 3$ or 4. We set $K_s^{(l)} = I_{s_0} \circ \cdots \circ I_{s_{n(l)-1}}(\overline{U_A})$. Then $K_s^{(l+1)} \subset K_s^{(l)}$ and

$$\Lambda^{-1}(s) = \bigcap_{l=1}^{\infty} K_s^{(l)}.$$

Note that $\bigcap_{l=1}^{\infty} K_s^{(l)}$ is a one-point set, which is the consequence of that I_k decreases the Poincaré distance on V_A or V_B, see [1].

We reconsider the sequence

$$\left\{ s^{(n)} = (\underbrace{1, 1, \ldots, 1}_{n \text{ times}}, \mathsf{B}, \mathsf{B}, \mathsf{B}, \ldots) \right\}_{n=0}^{\infty}$$

in Σ. It converges to $s = (1, 1, 1, \ldots) \notin \Sigma$ with respect to ρ. However it converges to $s = (\mathsf{A}, \mathsf{A}, \mathsf{A}, \ldots) \in \Sigma$ with respect to $\mathcal{O}$. We check that the convergence with respect to $\mathcal{O}$ is "appropriate". By the definition of Λ^{-1},

$$\Lambda^{-1}(s^{(n)}) = \underbrace{I_1 \circ \cdots \circ I_1}_{n \text{ times}}(K_B).$$

Let $K^{(n)} = \Lambda^{-1}(s^{(n)})$. Since I_1 decreases the Poincaré distance on V_A, the sequence $\{K^{(n)}\}_{n=0}^{\infty} \subset K(f)^*$ converges to not $K_A \in K(f)^*$ but a one-point set $K = \{\zeta\}$ with respect to the Hausdorff metric d_H. The point ζ is actually in ∂K_A, and therefore $K \notin K(f)^*$. We expect that $\{K^{(n)}\}_{n=0}^{\infty}$ converges to K_A with respect to $\mathcal{G}$. In fact,

$$\begin{aligned} \lim_{n\to\infty} K^{(n)} &= \lim_{n\to\infty} \Lambda^{-1}(s^{(n)}) = \Lambda^{-1}\Big(\lim_{n\to\infty} s^{(n)}\Big) \\ &= \Lambda^{-1}(s) = K_A \end{aligned}$$

since $\Lambda^{-1} : (\Sigma, \mathcal{O}) \to (K(f)^*, \mathcal{G})$ is continuous. Therefore we express that the convergence of $\{s^{(n)}\}_{n=0}^{\infty}$ with respect to $\mathcal{O}$ is "appropriate" in the sense that $\{K^{(n)}\}_{n=0}^{\infty}$ converges to K_A with respect to $\mathcal{G}$.

5. Applications. The following two theorems are fundamental.

Theorem 5.1 [3]. *Let g be a rational function of degree at least two. If $z \in J(g)$, then*

$$J(g) = \overline{\bigcup_{k=1}^{\infty} g^{-k}(z)}.$$

Theorem 5.2 [3]. *Let g be a rational function of degree at least two. Then*

$$J(g) = \overline{\{\text{repelling periodic point of } g\}}.$$

We obtain analogies of Theorem 5.1 and 5.2.

Theorem 5.3. *Let $(\Sigma, \mathcal{O})$ be as in Theorem 1.7 and let $s \in \Sigma$. Then*

$$\Sigma = \overline{\bigcup_{k=1}^{\infty} \sigma^{-k}(s)},$$

where the closure is taken in $(\Sigma, \mathcal{O})$.

Proof. Let $s = (s_0, s_1, s_2, \ldots) \in \Sigma$ and let

$$u = \begin{cases} 1 \text{ or } 2 & \text{if } s_0 \neq \mathsf{A}, \\ 3 \text{ or } 4 & \text{if } s_0 \neq \mathsf{B}. \end{cases}$$

Then $(u, s_0, s_1, s_2, \ldots) \in \sigma^{-1}(s)$. For $t = (\mathsf{A}, \mathsf{A}, \mathsf{A}, \ldots)$, we consider the sequence

$$\left\{ s^{(\alpha)} = (\underbrace{1, \ldots, 1}_{\alpha \text{ times}}, u, s_0, s_1, \ldots) \right\}_{\alpha=1}^{\infty} \subset \bigcup_{k=1}^{\infty} \sigma^{-k}(s).$$

Then $s^{(\alpha)}$ converges to $t = (\mathsf{A}, \mathsf{A}, \mathsf{A}, \ldots)$ with respect to $\mathcal{O}$. Next, for $t = (t_0, t_1, \ldots, t_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \ldots) \in \Sigma$ with $t_l \neq \mathsf{A}$, we consider the sequence

$$\left\{ s^{(\alpha)} = (t_0, \ldots, t_l, \underbrace{1, \ldots, 1}_{\alpha \text{ times}}, u, s_0, s_1, \ldots) \right\}_{\alpha=1}^{\infty} \subset \bigcup_{k=1}^{\infty} \sigma^{-k}(s).$$

Then $s^{(\alpha)}$ converges to $t = (t_0, t_1, \ldots, t_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \ldots)$ with respect to $\mathcal{O}$. In the case of "B", we choose "3" instead of "1". Finally, for $t = (t_0, t_1, t_2, \ldots) \in \Sigma \cap \Sigma_4$, we consider the sequence

$$\left\{ s^{(\alpha)} = (t_0, t_1, \ldots, t_\alpha, u, s_0, s_1, s_2, \ldots) \right\}_{\alpha=1}^{\infty} \subset \bigcup_{k=1}^{\infty} \sigma^{-k}(s).$$

Then $s^{(\alpha)}$ converges to $t = (t_0, t_1, t_2, \ldots)$ with respect to $\mathcal{O}$. □

Remark 5.4. The closure of the backward orbit of $s \in \Sigma$ under σ does not necessarily coincide with Σ in (Σ, ρ). For example,

$$(\mathsf{A}, \mathsf{A}, \mathsf{A}, \ldots) \notin \overline{\bigcup_{k=1}^{\infty} \sigma^{-k}((\mathsf{B}, \mathsf{B}, \mathsf{B}, \ldots))},$$

where the closure is taken in (Σ, ρ).

Corollary 5.5. *Let* $(K(f)^*, \mathcal{G})$ *be as in Corollary* 1.8 *and let* $K \in K(f)^*$. *Then*

$$K(f)^* = \overline{\bigcup_{k=1}^{\infty} F^{-k}(K)},$$

where the closure is taken in $(K(f)^*, \mathcal{G})$.

Theorem 5.6. *Let* $(\Sigma, \mathcal{O})$ *be as in Theorem* 1.7. *Then*

$$\Sigma = \overline{\{\text{periodic point of } \sigma \text{ in } \Sigma\}},$$

where the closure is taken in $(\Sigma, \mathcal{O})$.

Proof. We show that each non-periodic point $t \in \Sigma$ is a limit point of a sequence of periodic points of Σ. For $t = (t_0, t_1, \dots, t_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \dots) \in \Sigma$ with $t_l \neq \mathsf{A}$, we consider the sequence

$$\left\{ s^{(\alpha)} = (t_0, t_1, \dots, t_l, \underbrace{1, 1, \dots, 1}_{\alpha \text{ times}}, t_0, t_1, \dots, t_l, \underbrace{1, 1, \dots, 1}_{\alpha \text{ times}}, \dots) \right\}_{\alpha=1}^{\infty}$$

of period $\alpha + l + 1$. Then $s^{(\alpha)}$ converges to $t = (t_0, t_1, \dots, t_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$ with respect to $\mathcal{O}$. In the case of "B", we choose "3" instead of "1". For a non-periodic point $t = (t_0, t_1, t_2, \dots) \in \Sigma \cap \Sigma_4$, we consider the sequence

$$\left\{ s^{(\alpha)} = (t_0, t_1, \dots, t_\alpha, t_0, t_1, \dots, t_\alpha, \dots) \right\}_{\alpha=\beta}^{\infty}$$

of period $\alpha + 1$, where β is a natural number which satisfies $s^{(\beta)} \in \Sigma$. Then $s^{(\alpha)}$ converges to $t = (t_0, t_1, t_2, \dots)$ with respect to $\mathcal{O}$. $\square$

Remark 5.7. The closure of the set of all periodic points of Σ does not coincide with Σ in (Σ, ρ) since $t = (t_0, t_1, \dots, t_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$ with $t_l \neq \mathsf{A}$ is an isolated point in (Σ, ρ).

Corollary 5.8. *Let* $(K(f)^*, \mathcal{G})$ *be as in Corollary* 1.8. *Then*

$$K(f)^* = \overline{\{\text{periodic point of } F \text{ in } K(f)^*\}},$$

where the closure is taken in $(K(f)^*, \mathcal{G})$.

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