New proofs of the trace theorem of Sobolev spaces

By Yoichi MIYAZAKI

School of Dentistry, Nihon University, 1-8-13 Kanda-Surugadai,
Chiyoda-ku, Tokyo 101-8310, Japan

(Communicated by Heisuke HIRONAKA, M.J.A., June 10, 2008)

Abstract: We present three new proofs of the trace theorem of L_p Sobolev spaces, which do not rely on the theory of interpolation spaces. The first method originates in Morrey's proof for the Sobolev embedding theorem concerning the Hölder-Zygmund space. The second method is based on Muramatu's integral formula and the third method is based on an integral operator with Gauss kernel. These methods give unified viewpoints for the proofs of the trace theorem and the Sobolev embedding theorem.

Key words: Sobolev space; Besov space; trace theorem; Sobolev embedding theorem.

Introduction. The trace theorem of L_p Sobolev spaces $H_p^\sigma(\mathbf{R}^n)$ has an important role in the boundary value problems of partial differential equations. It is usually proved by the theory of interpolation spaces (see [1,2,8]). There are also elementary proofs which do not rely on the theory of interpolation spaces (see [3,4,7]).

In this paper, we present three new proofs of the trace theorem. The first method, for $\sigma = 1$, is direct and originates in Morrey's proof for the Sobolev embedding theorem concerning the Hölder-Zygmund space. The second method, for $\sigma = m$ with a positive integer m, is based on Muramatu's integral formula [6]. The third method, for general σ, is based on an integral operator with Gauss kernel.

In any method, we adopt the characterization by the difference operator as the definition of the Besove space $B_{pp}^\sigma(\mathbf{R}^n)$ with $\sigma > 0$ and $1 < p < \infty$, which is given below. Let σ be written as

$$\sigma = j + \tau, \quad j \in \mathbf{N}_0, \quad 0 < \tau \le 1,$$

and set

$$|u|_{B_{pp}^\tau(\mathbf{R}^n)} = \left(\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|\Delta_h^{1+[\tau]} u(x)|^p}{|h|^{\tau p + n}} \, dx \, dh \right)^{1/p},$$

$$|u|_{B_{pp}^\sigma(\mathbf{R}^n)} = \sum_{|\alpha| = j} |u^{(\alpha)}|_{B_{pp}^\tau(\mathbf{R}^n)},$$

where $\mathbf{N}_0$ is the set of non-negative integers, Δ_h is the difference operator given by $\Delta_h u(x) = u(x+h) - u(x)$ and $[\tau]$ denotes the largest integer not exceeding τ. Then the Besov space $B_{pp}^\sigma(\mathbf{R}^n)$ is defined by

$$B_{pp}^\sigma(\mathbf{R}^n) = \{ u \in H_p^j(\mathbf{R}^n) : |u|_{B_{pp}^\sigma(\mathbf{R}^n)} < \infty \},$$

and the norm is given by

$$\|u\|_{B_{pp}^\sigma(\mathbf{R}^n)} = \sum_{|\alpha| \le j} \|u^{(\alpha)}\|_{L_p(\mathbf{R}^n)} + |u|_{B_{pp}^\sigma(\mathbf{R}^n)}.$$

Usual modification for $p = \infty$ defines $B_{\infty\infty}^\sigma(\mathbf{R}^n)$, which is written as $\mathcal{C}^\sigma(\mathbf{R}^n)$.

The trace theorem is stated as follows:

Theorem 1. *Let n, k be integers with $n \ge 2$, $1 \le k < n$, and let $1 < p < \infty$ and $\sigma > k/p$. Let $x \in \mathbf{R}^n$ be written as $x = (x', x'')$ with $x' \in \mathbf{R}^{n-k}$ and $x'' \in \mathbf{R}^k$.*

Then the trace operator $\mathrm{Tr} : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^{n-k})$ *defined by*

$$(\mathrm{Tr}\, u)(x') = u(x', 0), \quad u \in \mathcal{S}(\mathbf{R}^n)$$

can be extended to a bounded linear operator $H_p^\sigma(\mathbf{R}^n) \to B_{pp}^{\sigma - k/p}(\mathbf{R}^{n-k})$. Namely, there exists a constant C depending only on n, k, p and σ such that

$$\|\mathrm{Tr}\, u\|_{B_{pp}^{\sigma-k/p}(\mathbf{R}^{n-k})} \le C \|u\|_{H_p^\sigma(\mathbf{R}^n)}. \tag{1}$$

In particular, if $\sigma = m$ with an integer $m > 0$ then

$$|\mathrm{Tr}\, u|_{B_{pp}^{m-k/p}(\mathbf{R}^n)} \le C \sum_{|\alpha| = m} \|u^{(\alpha)}\|_{L_p(\mathbf{R}^n)}. \tag{2}$$

Any of our methods to prove Theorem 1 also works for the proof of Theorem 2 below, which corresponds to the case $k = n$ in Theorem 1.

Theorem 2. *Let n be an integer with $n \ge 1$, and let $1 < p < \infty$ and $\sigma > n/p$.*

2000 Mathematics Subject Classification. Primary 46E35.

Then $H^\sigma_p(\mathbf{R}^n)$ is continuously embedded in $\mathcal{C}^{\sigma-n/p}(\mathbf{R}^n)$. Namely, there exists a constant C depending only on n, p and σ such that

$$\|u\|_{\mathcal{C}^{\sigma-n/p}(\mathbf{R}^n)} \le C\|u\|_{H^\sigma_p(\mathbf{R}^n)}. \tag{3}$$

In any method, the estimate of the Besov seminorm is reduced to the following lemma, which is a special case of boundedness theorem on integral operators.

Lemma 0.1. *Let k, l be positive integers, let $1 < p < \infty$, and let $K(s)$ be a measurable function of $s \in \mathbf{R}_+ = (0,\infty)$. Then the linear operator*

$$f \mapsto \int_{\mathbf{R}^k} K\left(\frac{|y|}{|h|}\right) f(y)\,\frac{dy}{|y|^k} \tag{4}$$

is bounded from $L_p(\mathbf{R}^k, |y|^{-k}dy)$ to $L_p(\mathbf{R}^l, |h|^{-l}dh)$ if K is in $L_1(\mathbf{R}_+, ds/s)$, and it is bounded from $L_p(\mathbf{R}^k, |y|^{-k}dy)$ to $L_\infty(\mathbf{R}^l)$ if K is in $L_q(\mathbf{R}_+, ds/s)$ with $p^{-1}+q^{-1}=1$.

In the following sections we denote by C various constants depending only on n, k, p and σ which may differ from line to line.

1. Application of Morrey's method. Let $\sigma = 1$. In order to provide insight into the first method, we review Morrey's proof [5, pp.79–80] of Theorem 2 briefly.

In the proof of Theorem 2 we have to obtain the estimate

$$|u(x+2h)-u(x)| \le C|h|^{1-n/p}, \quad h \in \mathbf{R}^n \tag{5}$$

for $u \in H^1_p(\mathbf{R}^n)$. To do so we consider the mean value of $u(x)$ on the ball of radius $|h|$ centered at $x+h$:

$$m_h(x) = \frac{1}{|B_h|}\int_{B_h} u(x+h+z)\,dz,$$

where B_h is the ball of radius $|h|$ centered at the origin and $|B_h|$ denotes the volume of B_h. Then we get (5) by evaluating

$$\begin{aligned} &m_h(x) - u(x)\\ &= \frac{1}{|B_h|}\int_{B_h} dz \int_0^1 \nabla u(x+sh+sz)\cdot(h+z)\,ds \end{aligned}$$

and $m_h(x) - u(x+2h)$.

If we note that the Sobolev embedding theorem is concerned with the restriction of a function to a point, 0-dimensional hyperplane, and that B_h is an n-dimensional ball, we can translate the idea of Morrey's method for the proof of Theorem 1 as follows:

"In order to estimate the restriction of a function to an $(n-k)$-dimensional hyperplane, consider the mean value of the function on a k-dimensional ball."

Proof of Theorem 1 for $\boldsymbol{\sigma = 1}$. It is sufficient to show (1) and (2) for $u \in \mathcal{S}(\mathbf{R}^n)$, since $\mathcal{S}(\mathbf{R}^n)$ is dense in $H^1_p(\mathbf{R}^n)$. We write a point $x \in \mathbf{R}^n$ as $x = (x', x'') \in \mathbf{R}^{n-k}\times\mathbf{R}^k$. For $h \in \mathbf{R}^{n-k}$ set $B_h = \{z'' \in \mathbf{R}^k : |z''| \le |h|\}$ and denote by $|B_h|$ the k-dimensional volume of B_h.

To evaluate $u(x'+2h,0) - u(x',0)$ we set

$$m_h(x') = \frac{1}{|B_h|}\int_{B_h} u(x'+h, z'')\,dz'' \tag{6}$$

and

$$\begin{aligned} F_h(x') &= m_h(x') - u(x',0)\\ &= \frac{1}{|B_h|}\int_{B_h} dz'' \int_0^1 \nabla u(x'+sh, sz'')\cdot(h,z'')\,ds. \end{aligned}$$

Using Minkowski's inequality, changing the variables with $y'' = sz''$, and writing $v(x'') = \|\nabla u(\,\cdot\,, x'')\|_{L_p(\mathbf{R}^{n-k})}$, we have

$$\begin{aligned} &\frac{\|F_h\|_{L_p(\mathbf{R}^{n-k})}}{|h|^{1-k/p}} \tag{7}\\ &\le C|h|^{-k+k/p}\int_0^1 ds \int_{B_h} \|\nabla u(\,\cdot\,, sz'')\|_{L_p(\mathbf{R}^{n-k})}\,dz''\\ &= C|h|^{-k+k/p}\int_0^1 s^{-k}ds \int_{|y''|\le s|h|} v(y'')\,dy''\\ &= C|h|^{-k+k/p}\int_{|y''|\le|h|}\left\{\int_{|y''|/|h|}^1 s^{-k}ds\right\} v(y'')\,dy''\\ &= C\int_{\mathbf{R}^k} \lambda_k\left(\frac{|y''|}{|h|}\right)\{|y''|^{k/p}v(y'')\}\,\frac{dy''}{|y''|^k}, \end{aligned}$$

where $\lambda_k(s) = s^{k-k/p}(s^{1-k}-1)/(k-1)$ for $0 < s \le 1$, $k > 1$, $\lambda_1(s) = s^{1-1/p}(-\log s)$ for $0 < s \le 1$, and $\lambda_k(s) = 0$ for $s > 1$, $k \ge 1$. Hence, Lemma 0.1 implies

$$\left\|\frac{\|F_h\|_{L_p(\mathbf{R}^{n-k})}}{|h|^{1-k/p}}\right\|_{L_p(\mathbf{R}^{n-k}, |h|^{k-n}dh)} \le C\|\nabla u\|_{L_p(\mathbf{R}^n)},$$

$$\sup_{h\in\mathbf{R}^{n-k}} \frac{\|F_h\|_{L_p(\mathbf{R}^{n-k})}}{|h|^{1-k/p}} \le C\|\nabla u\|_{L_p(\mathbf{R}^n)},$$

for $\lambda_k \in L_1(\mathbf{R}_+, ds/s)\cap L_\infty(\mathbf{R}_+)$.

We get a similar estimate for

$$G_h(x') := m_h(x') - u(x'+2h, 0).$$

So (2) follows from $u(x'+2h,0)-u(x',0) = F_h(x') - G_h(x')$.

We now evaluate the L_p norm of $u(x',0)$. We have already seen that $\|F_h\|_{L_p(\mathbf{R}^{n-k})} \le C|h|^{1-k/p}\|\nabla u\|_{L_p(\mathbf{R}^n)}$. On the other hand, we have by (6) and Hölder's inequality

$$\|m_h\|_{L_p(\mathbf{R}^{n-k})} \le C|h|^{-k}\int_{B_h}\|u(\,\cdot\,,y'')\|_{L_p(\mathbf{R}^{n-k})}\,dy'' \le C|h|^{-k/p}\|u\|_{L_p(\mathbf{R}^n)}.$$

Thus $u(x',0) = m_h(x') - F_h(x')$ with $|h| = 1$ gives

$$\|\mathrm{Tr}\,u\|_{L_p(\mathbf{R}^{n-k})} \le C\|u\|_{H^1_p(\mathbf{R}^n)}.$$

This combined with (2) yields (1). □

Remark 1.1. In the proof we can replace B_h by $B_h \cap (\mathbf{R}_+)^k$. Then we obtain

$$\|\mathrm{Tr}\,u\|_{B^{1-k/p}_{pp}(\mathbf{R}^{n-k})} \le C\|u\|_{H^1_p(\mathbf{R}^{n-k}\times(\mathbf{R}_+)^k)}.$$

Remark 1.2. Our first method is very similar to that of DiBenedetto [4] who proved Theorem 1 for $\sigma = 1$ and $k = 1$. He derived (2) by setting $m_h(x') = u(x'+h,|h|)$ instead of (6) and using the inequality

$$\|u(\,\cdot\,+h,|h|) - u(\,\cdot\,,0)\|_{L_p(\mathbf{R}^{n-1})} \le \sqrt{2}|h|\int_0^1\|\nabla u(\,\cdot\,,s|h|)\|_{L_p(\mathbf{R}^{n-1})}\,ds$$

and the identity

$$\Big\||h|^{1/p}\|\nabla u(\,\cdot\,,s|h|)\|_{L_p(\mathbf{R}^{n-1})}\Big\|_{L_p(\mathbf{R}^{n-1},|h|^{1-n}dh)} = (\omega_{n-1}s^{-1})^{1/p}\|\nabla u\|_{L_p(\mathbf{R}^{n-1}\times\mathbf{R}_+)},$$

which is obtained by integration in polar coordinates. Here ω_{n-1} denotes the surface area of the unit sphere in $\mathbf{R}^{n-1}$. Whereas his method requires another consideration to obtain $u(x',0) \in L_p(\mathbf{R}^{n-1})$, in our method this is an immediate consequence of m_h, $F_h \in L_p(\mathbf{R}^{n-1})$.

2. Method of Muramatu's formula. In this section we assume that m is a positive integer. Before giving the proof of Theorem 1 for $\sigma = m$ we briefly review Muramatu's formula, which expresses a function by its regularizations.

Choose a function $\rho \in C_0^\infty(\mathbf{R}^n)$ satisfying $\int_{\mathbf{R}^n}\rho(x)\,dx = 1$ and $\mathrm{supp}\,\rho \subset \{x \in \mathbf{R}^n : |x| < 1\}$. Let N be a positive integer and set

$$\omega(x) = \sum_{|\alpha|<N}\frac{1}{\alpha!}\partial_x^\alpha\{x^\alpha\rho(x)\},$$

$$M(x) = \sum_{|\alpha|=N}M_\alpha^{(\alpha)}(x), \quad M_\alpha(x) = \frac{N}{\alpha!}x^\alpha\rho(x).$$

For $t > 0$ we set $\omega_t(x) = t^{-n}\omega(x/t)$ and $M_t(x) = t^{-n}M(x/t)$. Using the relations $\partial_t\{\omega_t(x)\} = -t^{-1}M_t(x)$ and $\lim_{t\to+0}\omega_t * f(x) = f(x)$, we have

$$f(x) = \int_0^R M_t * f(x)\,\frac{dt}{t} + \omega_R * f(x), \quad R > 0 \tag{8}$$

for $f \in \mathcal{S}(\mathbf{R}^n)$. Taking the limit as $R \to \infty$, we also have

$$f(x) = \int_0^\infty M_t * f(x)\,\frac{dt}{t}. \tag{9}$$

We note that the relation

$$(M_\alpha^{(\alpha)})_t * f(x) = t^{|\beta|}(M_\alpha^{(\alpha-\beta)})_t * f^{(\beta)}(x)$$

holds for $0 < \beta \le \alpha$ by integration by parts.

In the argument below, it is convenient to assume that $\rho(x)$ is written in the form

$$\rho(x) = \rho_0(x')\rho_1(x'') \tag{10}$$

with $\rho_0 \in C_0^\infty(\mathbf{R}^{n-k})$ and $\rho_1 \in C_0^\infty(\mathbf{R}^k)$.

Proof of Theorem 1 for $\boldsymbol{\sigma = m}$. Let $u \in \mathcal{S}(\mathbf{R}^n)$ and let

$$m - k/p = j + \tau, \quad j \in \mathbf{N}_0, \quad 0 < \tau \le 1. \tag{11}$$

First we shall evaluate $\|u^{(\alpha)}(x',0)\|_{L_p(\mathbf{R}^{n-k})}$ for $\alpha = (\alpha',0) \in \mathbf{N}_0^{n-k}\times\mathbf{N}_0^k$ with $|\alpha| \le j$. Applying (8) with $N = m$, $R = 1$ and $f = u^{(\alpha)}$ ($|\alpha| \le j$) and using integration by parts, we have

$$u^{(\alpha)}(x) = \sum_{|\beta|=m}\int_0^1 t^{m-|\alpha|}(K_{\alpha\beta})_t * u^{(\beta)}(x)\,\frac{dt}{t} + \omega_1^{(\alpha)} * u(x)$$

with some functions $K_{\alpha\beta} \in C_0^\infty(\mathbf{R}^n)$. It is easily seen that

$$\|K_t * f(x',0)\|_{L_p(\mathbf{R}^{n-k})} \le \|K_t\|_{L_1(L_q)}\|f\|_{L_p(\mathbf{R}^n)} \le t^{-k/p}\|K\|_{L_1(L_q)}\|f\|_{L_p(\mathbf{R}^n)}$$

for $K \in C_0^\infty(\mathbf{R}^n)$, where $p^{-1} + q^{-1} = 1$ and $\|K\|_{L_1(L_q)} = \int_{\mathbf{R}^{n-k}}\|K(x',x'')\|_{L_q(\mathbf{R}^k_{x''})}\,dx'$. Hence

$$\begin{aligned}\|u^{(\alpha)}(\,\cdot\,,0)\|_{L_p(\mathbf{R}^{n-k})} &\le C\sum_{|\beta|=m}\int_0^1 t^{m-|\alpha|-k/p}\|u^{(\beta)}\|_{L_p(\mathbf{R}^n)}\,\frac{dt}{t} + \|\omega_1^{(\alpha)}\|_{L_1(L_q)}\|u\|_{L_p(\mathbf{R}^n)}\\ &\le C\|u\|_{H_p^m(\mathbf{R}^n)},\end{aligned} \tag{12}$$

where we used $m - |\alpha| - k/p = (j - |\alpha|) + \tau > 0$.

Next we shall evaluate the Besov seminorm of

$u^{(\alpha)}(x',0)$, assuming $\alpha=(\alpha',0)$ with $|\alpha|=j$. Applying (9) with $f=u^{(\alpha)}$ and using integration by parts, we have

$$u^{(\alpha)}(x)=\sum_{|\beta|=m}\int_0^\infty t^{m-j}(K_{\alpha\beta})_t*u^{(\beta)}(x)\,\frac{dt}{t}$$

with some functions $K_{\alpha\beta}\in C_0^\infty(\mathbf{R}^n)$. In view of (10) we have

$$\begin{aligned}&\|\Delta_h^{1+[\tau]}(K_{\alpha\beta})_t(\,\cdot\,,x'')\|_{L_1(\mathbf{R}^{n-k})}\\&\le C\eta\Big(\frac{|h|}{t}\Big)^{1+[\tau]}t^{-k}\chi\Big(\frac{|x''|}{t}\Big)\end{aligned}$$

for $h\in\mathbf{R}^{n-k}$, where $\chi(s)$ is a characteristic function of the interval $[0,1]$ and $\eta(s)=\min\{s,1\}$ for $s>0$. Hence, writing $v_\beta(x'')=\|u^{(\beta)}(\,\cdot\,,x'')\|_{L_p(\mathbf{R}^{n-k})}$, we have

$$\begin{aligned}&\frac{\|\Delta_h^{1+[\tau]}u^{(\alpha)}(\,\cdot\,,0)\|_{L_p(\mathbf{R}^{n-k})}}{|h|^\tau}\\&\le C\sum_{|\beta|=m}\int_0^\infty\frac{t^\mu}{|h|^\tau}\eta\Big(\frac{|h|}{t}\Big)^{1+[\tau]}\frac{dt}{t}\\&\qquad\times\int_{\mathbf{R}^k}\chi\Big(\frac{|y''|}{t}\Big)v_\beta(y'')\,dy''\\&=C\sum_{|\beta|=m}\int_{\mathbf{R}^k}K\Big(\frac{|y''|}{|h|}\Big)\{|y''|^{k/p}v_\beta(y'')\}\frac{dy''}{|y''|^k},\end{aligned}$$

where $K(s)=s^\tau\int_1^\infty t^\mu\eta(s^{-1}t^{-1})^{1+[\tau]}t^{-1}\,dt$ and $\mu=\tau-k+k/p$. Thus, inequality (2) follows from Lemma 0.1, for

$$\int_0^\infty\eta\Big(\frac{1}{st}\Big)^{1+[\tau]}s^\tau\frac{ds}{s}=\frac{1+[\tau]}{\tau(1+[\tau]-\tau)}t^{-\tau}.$$

Since $\mathcal{S}(\mathbf{R}^n)$ is dense in $H_p^m(\mathbf{R}^n)$, (12) with (2) yields Theorem 1 for $\sigma=m$. □

Remark 2.1. In [6] Muramatu proved Theorem 1 for $k=1$ and general σ, using the second integral formula, which is derived by iterating (8), and the characterization of $H_p^\sigma(\mathbf{R}^n)$ by $L_p(\mathbf{R}^n,L_2([0,1],t^{-1}dt))$. If σ is a positive integer, the proof is simplified as above.

3. Method of Gauss kernel. Let $\langle D\rangle^\sigma$ with $\sigma>0$ be the Fourier multiplier with symbol $\langle\xi\rangle^\sigma$, where $\langle\xi\rangle=\sqrt{1+|\xi|^2}$. As is well known, $\langle D\rangle^\sigma$ defines isomorphisms $H_p^\sigma(\mathbf{R}^n)\to L_p(\mathbf{R}^n)$ and $\mathcal{S}(\mathbf{R}^n)\to\mathcal{S}(\mathbf{R}^n)$. So if we set $f=\langle D\rangle^\sigma u$ for $u\in H_p^\sigma(\mathbf{R}^n)$ then $u=\langle D\rangle^{-\sigma}f$ and $f\in L_p(\mathbf{R}^n)$ with $\|f\|_{L_p(\mathbf{R}^n)}\le C\|u\|_{H_p^\sigma(\mathbf{R}^n)}$. In particular, if $u\in\mathcal{S}(\mathbf{R}^n)$ then

$$u(x)=(2\pi)^{-n}\int_{\mathbf{R}^n}e^{ix\xi}\langle\xi\rangle^{-\sigma}\mathcal{F}f(\xi)\,d\xi \tag{13}$$

with $f\in\mathcal{S}(\mathbf{R}^n)$, where $\mathcal{F}f$ stands for the Fourier transform of f, i.e., $\mathcal{F}f(\xi)=\int_{\mathbf{R}^n}e^{-ix\xi}f(x)\,dx$.

For an integer $l\ge1$ we set

$$G_l(w)=(4\pi)^{-l/2}e^{-|w|^2/4},\quad G_{l,t}(w)=t^{-l}G_l(w/t)$$

for $t>0$, $w=(w_1,\dots,w_l)\in\mathbf{R}^l$. Then the third method starts with formula (14) below.

Lemma 3.1. *Let $u\in\mathcal{S}(\mathbf{R}^n)$ and $f\in\mathcal{S}(\mathbf{R}^n)$ satisfy $u=\langle D\rangle^{-\sigma}f$ with $\sigma>0$. Then*

$$u(x)=c_\sigma\int_0^\infty t^{\sigma-1}e^{-t^2}\,dt\int_{\mathbf{R}^n}G_{n,t}(x-y)f(y)\,dy \tag{14}$$

with $c_\sigma=2\Gamma(\sigma/2)^{-1}$.

Proof. The lemma follows from substitution of

$$\langle\xi\rangle^{-\sigma}=c_\sigma\int_0^\infty t^{\sigma-1}e^{-t^2\langle\xi\rangle^2}\,dt$$

in (13), Fubini's theorem and the formula of the Fourier transform of $e^{-|x|^2}$. □

Proof of Theorem 1 for general $\boldsymbol{\sigma}$. Let $u\in\mathcal{S}(\mathbf{R}^n)$ and let

$$\sigma-k/p=j+\tau,\quad j\in\mathbf{N}_0,\quad 0<\tau\le1.$$

First we shall evaluate $\|u^{(\alpha)}(x',0)\|_{L_p(\mathbf{R}^{n-k})}$ for $\alpha=(\alpha',0)\in\mathbf{N}_0^{n-k}\times\mathbf{N}_0^k$ with $|\alpha|\le j$. Replacing u and f by $u^{(\alpha)}$ and $f^{(\alpha)}$, respectively, in (14) and integrating by parts, we have

$$\begin{aligned}u^{(\alpha)}(x',0)&=c_\sigma\int_0^\infty t^{\sigma-1}e^{-t^2}\,dt\\&\quad\times\int_{\mathbf{R}^n}G_{n-k,t}^{(\alpha')}(x'-y')G_{k,t}(y'')f(y',y'')\,dy'\,dy''.\end{aligned}\tag{15}$$

Since $\|G_{n-k,t}^{(\alpha')}\|_{L_1(\mathbf{R}^{n-k})}\le Ct^{-|\alpha|}$ and $\|G_{k,t}\|_{L_q(\mathbf{R}^k)}=Ct^{-k/p}$ with $p^{-1}+q^{-1}=1$, we have

$$\begin{aligned}&\|u^{(\alpha)}(\,\cdot\,,0)\|_{L_p(\mathbf{R}^{n-k})}\\&\le C\|f\|_{L_p(\mathbf{R}^n)}\int_0^\infty t^{\sigma-|\alpha|-(k/p)-1}e^{-t^2}\,dt\\&\le C\|f\|_{L_p(\mathbf{R}^n)},\end{aligned}\tag{16}$$

where we used $\sigma-|\alpha|-k/p=(j-|\alpha|)+\tau>0$.

Next we shall evaluate the Besov seminorm of $u^{(\alpha)}(x',0)$, assuming $\alpha=(\alpha',0)$ with $|\alpha|=j$. Since

$$\|\Delta_h^{1+[\tau]}G_{n-k,t}^{(\alpha')}\|_{L_1(\mathbf{R}^{n-k})}\le Ct^{-j}\eta\Big(\frac{|h|}{t}\Big)^{1+[\tau]}$$

for $h\in\mathbf{R}^{n-k}$, (15) gives, with $v(x'')=\|f(\,\cdot\,,x'')\|_{L_p(\mathbf{R}^{n-k})}$, that

$$\frac{\|\Delta_h^{1+[\tau]} u^{(\alpha)}(\,\cdot\,,0)\|_{L_p(\mathbf{R}^{n-k})}}{|h|^\tau}$$
$$\leq C\int_0^\infty \frac{t^\mu}{|h|^\tau}\eta\Big(\frac{|h|}{t}\Big)^{1+[\tau]}\frac{dt}{t}$$
$$\times \int_{\mathbf{R}^k}\exp\Big(-\frac{|y''|^2}{4t^2}\Big)v(y'')\,dy''$$
$$= C\int_{\mathbf{R}^k} K\Big(\frac{|y''|}{|h|}\Big)\{|y''|^{k/p}v(y'')\}\frac{dy''}{|y''|^k},$$

where $\mu = \tau - k + k/p$ and $K(s) = s^\tau \int_0^\infty t^\mu e^{-1/(4t^2)}\eta(s^{-1}t^{-1})^{1+[\tau]}t^{-1}dt$. Therefore, applying Lemma 0.1, we have

$$|\mathrm{Tr}\,u|_{B_{pp}^{\sigma-k/p}(\mathbf{R}^{n-k})} \leq C\|f\|_{L_p(\mathbf{R}^n)}. \tag{17}$$

Since $\mathcal{S}(\mathbf{R}^n)$ is dense in $H_p^\sigma(\mathbf{R}^n)$, (16) and (17) yield Theorem 1. □

Remark 3.2. The method of Gauss kernel is in the same line as Stein's proof in [7], where he used the formula

$$u(x) = \int_{\mathbf{R}^n} K(x-y)f(y)\,dy$$

with $K(x) = (2\pi)^{-n}\int_{\mathbf{R}^n} e^{ix\xi}\langle\xi\rangle^{-\sigma}\,d\xi$, which is associated with Bessel functions. In our method we can avoid the difficulty of handling the singularity of the kernel $K(x)$.

As stated in the Introduction, our methods also work for the proof of Theorem 2. Here we give it by the method of Gauss kernel.

Proof of Theorem 2. We have only to show (3) for $u \in \mathcal{S}(\mathbf{R}^n)$. Let σ be written as

$$\sigma - n/p = j + \tau, \quad j \in \mathbf{N}_0, \quad 0 < \tau \leq 1.$$

First let $|\alpha| \leq j$. Then a formula similar to (15) and $\|G_{n,t}^{(\alpha)}\|_{L_q(\mathbf{R}^n)} \leq Ct^{-|\alpha|-n/p}$ with $p^{-1}+q^{-1}=1$ give

$$\begin{aligned} &|u^{(\alpha)}(x)| \\ &\leq C\|f\|_{L_p(\mathbf{R}^n)}\int_0^\infty t^{\sigma-|\alpha|-(n/p)-1}e^{-t^2}\,dt \\ &\leq C\|f\|_{L_p(\mathbf{R}^n)}, \end{aligned} \tag{18}$$

since $\sigma - |\alpha| - n/p = (j-|\alpha|)+\tau > 0$.

Next let $|\alpha| = j$. Then a formula similar to (15) and

$$\|\Delta_h^{1+[\tau]}G_{n,t}^{(\alpha)}\|_{L_q(\mathbf{R}^n)} \leq Ct^{-j-n/p}\eta\Big(\frac{|h|}{t}\Big)^{1+[\tau]}$$

give

$$\begin{aligned} &\frac{|\Delta_h^{1+[\tau]}u^{(\alpha)}(x)|}{|h|^\tau} \\ &\leq C\|f\|_{L_p(\mathbf{R}^n)}\int_0^\infty \Big(\frac{t}{|h|}\Big)^\tau \eta\Big(\frac{|h|}{t}\Big)^{1+[\tau]}\frac{dt}{t} \\ &\leq C\|f\|_{L_p(\mathbf{R}^n)}. \end{aligned} \tag{19}$$

Thus (3) follows from (18) and (19). □

Acknowledgements. The author wishes to thank the referee for the valuable comments which particularly simplified the proofs of Theorem 1 by regarding an integral operator

$$\int_0^\infty H_0\Big(\frac{t}{|h|}\Big)\frac{dt}{t}\int_{\mathbf{R}^k}H_1\Big(\frac{|y''|}{t}\Big)V(y'')\frac{dy''}{|y''|^k}$$

with $H_0,\,H_1 \in L_1(\mathbf{R}_+, ds/s)$ not as a composition of two bounded linear operators

$$L_p(\mathbf{R}^k, \tfrac{dx''}{|x''|^k}) \to L_p(\mathbf{R}_+, \tfrac{dt}{t}) \to L_p(\mathbf{R}^{n-k}, \tfrac{dh}{|h|^{n-k}})$$

but as a bounded linear operator

$$V \mapsto \int_{\mathbf{R}^k}K\Big(\frac{|y''|}{|h|}\Big)V(y'')\frac{dy''}{|y''|^k}$$

with $K(s) = \int_0^\infty H_0(st)H_1(t^{-1})t^{-1}dt$.

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