Proper actions of $SL(2, \mathbf{C})$ on irreducible complex symmetric spaces

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Abstract: We classify irreducible complex symmetric spaces that admit proper $SL(2, \mathbf{C})$-actions.*

Key words: Proper action; symmetric space; complex manifold; properly discontinuous action; Fuchs group; weighted Dynkin diagram.

1. Introduction and statement of main results. An affine manifold is said to be *locally symmetric* if its torsion tensors vanishes and the curvature tensor is invariant under all parallel translations. It is *complete* if any geodesic is defined for all time intervals. A basic problem is

Problem 1.1 (*See* [10]). *What discrete groups can arise as the fundamental groups of complete, locally symmetric spaces?*

Any complete, locally symmetric space M is represented as the Clifford–Klein form $\Gamma\backslash G/H$ where G/H is a globally symmetric space and Γ is a discrete subgroup of a Lie group G such that $\Gamma \simeq \pi_1(M)$. Consequently, Problem 1.1 may be reformalized as

Problem 1.2. *What discrete subgroups of G can act as discontinuous groups for G/H?*

Here, we say a discrete subgroup Γ of G is a *discontinuous group* for the homogeneous space G/H if the natural action of Γ on G/H is properly discontinuous and free. Particularly interesting is the case where H is non-compact, and where G/H carries a G-invariant non-Riemanian geometric structure. In this case, properly discontinuity of the Γ-action on G/H is much stronger than the discreteness of Γ in G. Problem 1.2 is difficult even for Lorentz symmetric spaces as was shown by the Calabi-Markus phenomenon [2] for relativistic spherical space forms, and also by Margulis' counter example [16] in $\mathbf{R}^3 = \mathbf{R}^{2,1}$ to Milnor's conjecture [19] for Lorentz flat cases. A systematic study of Problem 1.2 for the general was started in the late 1980*s* by Kobayashi: [4] followed by [2,5,6,14,15,17,20,22]. See [11,13,18] for surveys of the theory of discontinuous groups for non-Riemannian homogeneous spaces G/H developed during the last two decades.

In this paper, we focus on discontinuous groups for irreducible complex symmetric spaces. To be more precise, we consider the following

Setting 1.3. *$G_{\mathbf{C}}$ is a connected, complex simple Lie group, θ is a holomorphic involutive automorphism of $G_{\mathbf{C}}$, and $K_{\mathbf{C}}$ is an open subgroup of $G_{\mathbf{C}}^{\theta} := \{g \in G_{\mathbf{C}} : \theta g = g\}$.*

Then $G_{\mathbf{C}}/K_{\mathbf{C}}$ is an irreducible complex symmetric space. Classic examples include:

$$(G_{\mathbf{C}}, K_{\mathbf{C}}) = (SL(n, \mathbf{C}), SO(n, \mathbf{C})) \text{ where } \theta g = {}^t g^{-1}.$$

Then, our main results are stated as follows:

Theorem 1.4. *Let $G_{\mathbf{C}}$ be a complex simple Lie group, and $G_{\mathbf{C}}/K_{\mathbf{C}}$ a complex symmetric space. Then, the following four conditions are equivalent*:

i) *$G_{\mathbf{C}}/K_{\mathbf{C}}$ admits an infinite discontinuous group generated by a unipotent element of $G_{\mathbf{C}}$.*
ii) *There exists a group homomorphism $\rho : SL(2, \mathbf{R}) \to G_{\mathbf{C}}$ such that $SL(2, \mathbf{R})$ acts properly on $G_{\mathbf{C}}/K_{\mathbf{C}}$ via ρ.*
iii) *There exists a holomorphic group homomorphism $\rho_{\mathbf{C}} : SL(2, \mathbf{C}) \to G_{\mathbf{C}}$ such that $SL(2, \mathbf{C})$ acts properly on $G_{\mathbf{C}}/K_{\mathbf{C}}$ via $\rho_{\mathbf{C}}$.*
iv) *The pair of Lie algebras $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{k}_{\mathbf{C}})$ is isomorphic to $(\mathfrak{so}(p+q, \mathbf{C}), \mathfrak{so}(p, \mathbf{C}) \oplus \mathfrak{so}(q, \mathbf{C}))$ (p, q is odd, $p + q \equiv 0 \mod 4$).*

A Kleinian group is a discrete subgroup of $SL(2, \mathbf{C})$. Any discrete subgroup of proper transformation groups is automatically a discontinuous group. Hence, we have a 'geometric model' of a Kleinian group on locally complex symmetric spaces as follows:

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Corollary 1.5. *For any Kleinian group* Γ*, and for any odd integers* p, q *such that* $p + q \equiv 0 \mod 4$*, there exists a locally complex symmetric space* X *whose fundamental group is isomorphic to* Γ*, and whose universal covering is biholomorphic to the global complex symmetric space* $\mathrm{Spin}(p+q, \mathbf{C})/Spin(p, \mathbf{C}) \times Spin(q, \mathbf{C})$.

Here, we note that $Spin(n, \mathbf{C})$ is a two-fold covering of $SO(n, \mathbf{C})$ if $n \geq 3$.

Our main results should be compared with two extreme cases: the existence problem of infinite discontinuous groups [2,4] and the existence problem of cocompact discontinuous groups [1,4–6,12,14,15,17,20,22].

First, the criterion [4] for the Calabi-Markus phenomenon says that there exist infinite discontinuous groups for $G_{\mathbf{C}}/K_{\mathbf{C}}$ if and only if $\mathrm{rank}_{\mathbf{R}}\, G_{\mathbf{C}} > \mathrm{rank}_{\mathbf{R}}\, K_{\mathbf{C}}$. Hence, we have

Fact 1.6 (*Calabi–Markus phenomenon*)**.** *An irreducible complex symmetric space* $G_{\mathbf{C}}/K_{\mathbf{C}}$ *admits an infinite discontinuous group if and only if* $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{k}_{\mathbf{C}})$ *is one of the following symmetric pairs*:

$(\mathfrak{sl}(n, \mathbf{C}), \mathfrak{so}(n, \mathbf{C}))$, $(\mathfrak{sl}(2n, \mathbf{C}), \mathfrak{sp}(n, \mathbf{C}))$, $(\mathfrak{so}(p+q, \mathbf{C}), \mathfrak{so}(p, \mathbf{C}) \oplus \mathfrak{so}(q, \mathbf{C}))$ (*pq is odd*)

$(\mathfrak{e}_{6,\mathbf{C}}, \mathfrak{sp}(4, \mathbf{C}))$, $(\mathfrak{e}_{6,\mathbf{C}}, \mathfrak{f}_{4,\mathbf{C}})$.

All the other irreducible complex symmetric spaces $G_{\mathbf{C}}/K_{\mathbf{C}}$ such as $SO(p+q, \mathbf{C})/SO(p, \mathbf{C}) \times SO(q, \mathbf{C})$ (pq is even) admit only finite discontinuous groups.

Second, let us consider the existence problem of cocompact discontinuous groups [10,18]. Here is the state-of-the-art in the case of complex symmetric spaces.

Fact 1.7 (*Existence of compact locally symmetric spaces*)**.**

(a) (*Kobayashi* [6] *and Benoist* [1]) *An irreducible complex symmetric space* $G_{\mathbf{C}}/K_{\mathbf{C}}$ *admits a cocompact discontinuous group only if* $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{k}_{\mathbf{C}})$ *is isomorphic to* $(\mathfrak{so}(4n, \mathbf{C}), \mathfrak{so}(4n-1, \mathbf{C}))$.

(b) (*Kobayashi–Yoshino* [12]) *The complex sphere* $SO(4n, \mathbf{C})/SO(4n-1, \mathbf{C})$ *admits a cocompact discontinuous group if* $n \leq 2$.

It is not known whether or not the complex sphere $SO(4n, \mathbf{C})/SO(4n-1, \mathbf{C})$ $(n \geq 3)$ admits a cocompact discontinuous group (See [12, Conjecture 2.4.4]).

2. Cartan projection and proper actions. We begin with a 'coarse geometry' of properly discontinuous actions introduced by Kobayashi:

Definition 2.1 [9, Definition 2.1.1]**.** For two subsets H, H' in a locally compact group G, we write $H \sim H'$ if there exists a compact subset S in G such that $H \subset SH'S$ and $H' \subset SHS$. We write $H \pitchfork H'$ if the closure of $H \cap SH'S$ is compact for any compact subset S in G.

As explained in [13, Section 3], the relation $\pitchfork$ generalizes the concept of properness of a group action. In other words, to understand whether an action is proper, or properly discontinuous, it is enough to understand the relation $\pitchfork$. In fact, if H and L are closed subgroups of G then we have:

The natural L-action on G/H is proper $\iff L \pitchfork H$.

The use of $\sim$ provides economies in considering the relation $\pitchfork$. In fact,

if $H \sim H'$ then $L \pitchfork H \iff L \pitchfork H'$.

For a reductive linear Lie group G, we have a Cartan decomposition $G = K \exp \mathfrak{a} K$, where $\mathfrak{a}$ is a maximally split abelian subspace in the Lie algebra $\mathfrak{g}$ of G. We write log for the inverse map of $\exp : \mathfrak{a} \to \exp \mathfrak{a}$. Let W be the Weyl group for the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. For a subset S of $G_{\mathbf{C}}$, we define an W-invariant set by

$$\mathfrak{a}(S) := \log(KSK \cap \exp \mathfrak{a}). \tag{1}$$

We note that $\mathfrak{a}(S) = W \cdot \log(S)$ if $S \subset \exp \mathfrak{a}$. Then, the following criterion of the relation $\pitchfork$ and $\sim$ holds.

Fact 2.2 ([4], [9, Theorems 3.4 and 5.6])**.**

(a) $H \sim H'$ *in* $G \iff \mathfrak{a}(H) \sim \mathfrak{a}(L)$ *in* $\mathfrak{a}$.

(b) $H \pitchfork H'$ *in* $G \iff \mathfrak{a}(H) \pitchfork \mathfrak{a}(L)$ *in* $\mathfrak{a}$.

In particular, the equivalence (1) implies $\mathfrak{a}(H) \sim \mathfrak{a}(gHg^{-1})$ for any $g \in G$.

Remark 2.3. Our main results can be proved by using only the criterion of proper actions of reductive subgroups given in [4]. However we use a more general criterion given in [9] mainly because the concept $\sim$ and $\pitchfork$ simplifies the exposition of our proof.

3. The Calabi–Markus phenomenon for $G_{\mathbf{C}}/K_{\mathbf{C}}$. Let $\mathrm{Aut}(\mathfrak{g}_{\mathbf{C}})$ be the group of Lie algebra automorphisms of the complex simple Lie algebra $\mathfrak{g}_{\mathbf{C}}$. We write $\mathrm{Int}(\mathfrak{g}_{\mathbf{C}})$ for the group of inner automorphisms, namely, the group generated by $e^{\mathrm{ad}(X)}$ $(X \in \mathfrak{g}_{\mathbf{C}})$. Then, $\mathrm{Int}(\mathfrak{g}_{\mathbf{C}})$ is the identity component of $\mathrm{Aut}(\mathfrak{g}_{\mathbf{C}})$. If $G_{\mathbf{C}}$ is a connected Lie group with Lie algebra $\mathfrak{g}_{\mathbf{C}}$, the adjoint representation $\mathrm{Ad} : G_{\mathbf{C}} \to \mathrm{Int}(\mathfrak{g}_{\mathbf{C}})$ is surjective and its kernel is a finite group. An element of $\mathrm{Aut}(\mathfrak{g}_{\mathbf{C}}) \setminus \mathrm{Int}(\mathfrak{g}_{\mathbf{C}})$ is called an *outer automorphism.*

Lemma 3.1. *If $G_{\mathbf{C}}/K_{\mathbf{C}}$ admits an infinite discontinuous group, then θ is an outer automorphism. In particular, $\mathfrak{g}_{\mathbf{C}}$ is one of $\mathfrak{sl}(n, \mathbf{C})$, $\mathfrak{so}(2n, \mathbf{C})$, or $\mathfrak{e}_{6,\mathbf{C}}$.*

Proof. If θ were an inner automorphism, then $K_{\mathbf{C}}$ would have the equal rank with $G_{\mathbf{C}}$ because $\mathrm{Ad}(K_{\mathbf{C}})$ is the centralizer of the elliptic element θ in $\mathrm{Int}(\mathfrak{g}_{\mathbf{C}}) = \mathrm{Ad}(G_{\mathbf{C}})$. In turn, we would have $\mathrm{rank}_{\mathbf{R}} K_{\mathbf{C}} = \mathrm{rank}_{\mathbf{R}} G_{\mathbf{C}}$, which is equivalent to the condition that $G_{\mathbf{C}}/K_{\mathbf{C}}$ does not admit an infinite discontinuous group by [4, Corollary 4.4]. Hence, the lemma is proved. □

Fact 1.6 gives the complete list of the complex symmetric pairs $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{k}_{\mathbf{C}})$ such that $G_{\mathbf{C}}/K_{\mathbf{C}}$ does not admit an infinite discontinuous group.

4. Outer automorphisms and nilpotent orbits. An element $X \in \mathfrak{g}_{\mathbf{C}}$ is called *nilpotent* if $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g}_{\mathbf{C}})$ is a nilpotent endomorphism. The set $\mathcal{N}$ of nilpotent elements in $\mathfrak{g}_{\mathbf{C}}$ is a $G_{\mathbf{C}}$-invariant algebraic variety. There are finitely many $G_{\mathbf{C}}$-orbits in $\mathcal{N}$. The adjoint orbit $\mathrm{Ad}(G_{\mathbf{C}})X \in \mathcal{N}$ is said to be a *nilpotent orbit.* The group $\mathrm{Aut}(\mathfrak{g}_{\mathbf{C}})$ normalizes the identity component $\mathrm{Int}(\mathfrak{g}_{\mathbf{C}})$. That is, if $\tau \in \mathrm{Aut}(\mathfrak{g}_{\mathbf{C}})$, then τ acts on $\mathrm{Int}(\mathfrak{g}_{\mathbf{C}})$ by $A \to \tau \circ A \circ \tau^{-1}$. In particular, τ sends each nilpotent orbit to a (possibly different) nilpotent orbit because

$$\begin{aligned}\tau(\mathrm{Ad}(G_{\mathbf{C}})X) = \tau(\mathrm{Int}(\mathfrak{g}_{\mathbf{C}})X) &= \mathrm{Int}(\mathfrak{g}_{\mathbf{C}})\tau X\\ &= \mathrm{Ad}(G_{\mathbf{C}})\tau X.\end{aligned}$$

With regard to outer automorphisms we have:

Theorem 4.1. *Let $\mathfrak{g}_{\mathbf{C}}$ be a complex simple Lie algebra. Then, the following two conditions on $\mathfrak{g}_{\mathbf{C}}$ are equivalent*:

(i) *Any outer automorphism leaves every nilpotent orbit invariant.*

(ii) *$\mathfrak{g}_{\mathbf{C}}$ is not isomorphic to $\mathfrak{so}(4n, \mathbf{C})$.*

Sketch of proof. First of all, we recall the Dynkin–Kostant theory that describes nilpotent orbits by means of weighted Dynkin diagrams. We fix a Cartan subalgebra $\mathfrak{h}_{\mathbf{C}}$ of $\mathfrak{g}_{\mathbf{C}}$, and a positive root system $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. We denote by Ψ the set of simple roots, and by $\mathfrak{h}_{\mathbf{C}}^+$ the dominant chamber. We set elements in $\mathfrak{sl}(2, \mathbf{R})$ by $h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Suppose $X \in \mathcal{N}$. Then, there exists a Lie algebra homomorphism $\rho : \mathfrak{sl}(2, \mathbf{R}) \to \mathfrak{g}_{\mathbf{C}}$ such that $\rho(e) = X$ by the Jacobson–Morosov theorem. Since $\rho(h)$ is a semisimple element, there exists a unique element in $\mathfrak{h}_{\mathbf{C}}^+$, denoted by H, which is conjugate to $\rho(h)$. We set

$$F_X(\alpha) := \alpha(H) \quad \text{for } \alpha \in \Psi.$$

Then $F_X(\alpha) \in \{0, 1, 2\}$ for any $\alpha \in \Psi$, and $F_X \equiv F_{X'}$ iff X is conjugate to X' by $G_{\mathbf{C}}$. Hence, the injective map

$$\mathcal{N}/\,\mathrm{Ad}(G_{\mathbf{C}}) \to \mathrm{Map}(\Psi, \{0, 1, 2\}),\ \mathrm{Ad}(G_{\mathbf{C}})X \mapsto F_X$$

classifies the set of nilpotent orbits. The map F_X is represented by the *weighted Dynkin diagram* (See [3]).

Let τ be an automorphism of $\mathfrak{g}_{\mathbf{C}}$. Then, there exists an inner automorphism σ of $\mathfrak{g}_{\mathbf{C}}$ such that $\sigma\tau(\mathfrak{h}_{\mathbf{C}}) = \mathfrak{h}_{\mathbf{C}}$ and $(\sigma\tau)^*(\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})) = \Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. Here, $(\sigma\tau)^* : \mathfrak{h}_{\mathbf{C}}^* \to \mathfrak{h}_{\mathbf{C}}^*$ denotes the pull-back of $\sigma\tau|_{\mathfrak{h}_{\mathbf{C}}}$. Then $(\sigma\tau)^*$ induces an automorphism, to be denoted $\bar{\tau}$, of the Dynkin diagram. Then, $\bar{\tau}$ is independent of the choice of an inner automorphism σ. Then, we have:

Lemma 4.2. *For $X \in \mathcal{N}$, X and τX is conjugate by $G_{\mathbf{C}}$ if and only if its weighted Dynkin diagram F_X is invariant by $\bar{\tau}$.*

Thus, the condition (i) in Theorem 4.1 is equivalent to each of the following condition (iii) and (iv):

(iii) *For any Lie algebra homomorphism $\rho : \mathfrak{sl}(2, \mathbf{R}) \to \mathfrak{g}_{\mathbf{C}}$, $\tau\rho(e)$ is conjugate to $\rho(e)$ under* $\mathrm{Int}(\mathfrak{g}_{\mathbf{C}})$ *for any automorphism τ of $\mathfrak{g}_{\mathbf{C}}$.*

(iv) *For any nilpotent orbit in $\mathfrak{g}_{\mathbf{C}}$ the corresponding weighted Dynkin diagram is invariant by outer automorphisms of the Dynkin diagram.*

We can prove the quivalence (iv) and (ii) of Theorem 4.1 by using the classification of nilpotent orbits (see [3]). We should point out that the nilpotent orbit corresponding to the partition $[3^2, 1^2]$ for $SO(8, \mathbf{C})$ is missing in the table of Collingwood-McGovern [3, Example 5.3.7]. □

Suppose we are now in Setting 1.3. We return to the setting of Section 1. We take a maximal compact subgroup G_U of $G_{\mathbf{C}}$ such that $K_U := G_U \cap K_{\mathbf{C}}$ is a maximal compact subgroup of $K_{\mathbf{C}}$. Take a maximal torus of K_U, and extend it to a maximal torus H of G_U. We shall use the lower German letter to denote the Lie algebra, and use the subscript $\mathbf{C}$ to denote its complexification. Then, the complexified Lie algebra $\mathfrak{h}_{\mathbf{C}}$ of H is the Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$. In turn, $\mathfrak{a} := \mathfrak{h}_{\mathbf{C}} \cap \sqrt{-1}\mathfrak{g}_U$ is a maximal abelian subspace in $\sqrt{-1}\mathfrak{g}_U$, and $\mathfrak{a}^\theta$ is a maximal abelian subspace of $\sqrt{-1}\mathfrak{k}_U$. Since $\mathfrak{g}_{\mathbf{C}} =$

$\mathfrak{g}_U + \sqrt{-1}\mathfrak{g}_U$ is a Cartan decomposition, the subspace $\mathfrak{a}$ becomes a maximal split abelian subspace of the Lie algebra $\mathfrak{g}_{\mathbf{C}}$ (regarded as a real semisimple Lie algebra), and we have a Cartan decomposition $G_{\mathbf{C}} = G_U \exp \mathfrak{a}\, G_U$.

Lemma 4.3. $\mathfrak{a}(K_{\mathbf{C}}) = W \cdot \mathfrak{a}^\theta$.

Further, we can choose a positive system $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ such that $\theta^*\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}) = \Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. Then, θ induces an automorphism $\bar{\theta}$ of the Dynkin diagram. (We note that $\bar{\theta} = \mathrm{id}$ iff θ is an inner automorphism.) We have:

Corollary 4.4. *Let $\mathfrak{g}$ be a complex simple Lie algebra which is not isomorphic to $\mathfrak{so}(4n, \mathbf{C})$. Then, for any homomorphism $\rho:\ \mathfrak{sl}(2,\mathbf{R}) \to \mathfrak{g}_{\mathbf{C}}$,*

$$\mathfrak{a}(\rho(\mathfrak{sl}(2,\mathbf{R}))) \subset W \cdot \mathfrak{a}^\theta \textit{up to} \sim .$$

Proof. By Theorem 4.1, if $H \in \mathfrak{h}^+_{\mathbf{C}}$ is conjugate to $\rho(h)$, then $\theta H = H$. Since

$$\mathfrak{a}(\rho(\mathfrak{sl}(2,\mathbf{R}))) \sim \mathfrak{a}(\rho(\mathbf{R}h)) \sim \mathfrak{a}(\mathbf{R}H),$$

we get Corollary. □

5. Actions on $SO(p+q,\mathbf{C})/SO(p,\mathbf{C}) \times SO(q,\mathbf{C})$. The first step of the proof of Theorem 1.4 is to show:

Theorem 5.1. *Suppose we are in Setting 1.3. Let $\rho:\ \mathfrak{sl}(2,\mathbf{R}) \to \mathfrak{g}_{\mathbf{C}}$ be a Lie algebra homomorphism, and we set $X := \rho(e)$. We also use the same letter ρ to denote its lift to a group homomorphism $SL(2,\mathbf{R}) \to G_{\mathbf{C}}$. Then the following two conditions on the triple $(G_{\mathbf{C}}, K_{\mathbf{C}}, \rho)$ are equivalent:*

i) *$SL(2,\mathbf{R})$ acts properly on $G_{\mathbf{C}}/K_{\mathbf{C}}$ via ρ.*
ii) *The nilpotent orbit $\mathrm{Ad}(G_{\mathbf{C}})X$ is not θ-invariant.*

Proof. For $g \in G_{\mathbf{C}}$, we define $\rho^g(x) := g\rho(x)g^{-1}$. Then both of the conditions (i) and (ii) do not change if we replace ρ by ρ^g. Therefore, we can and do assume that

$$H := \rho(h) \in \mathfrak{a}_+.$$

Then, by the Dynkin–Kostant theory, the condition (ii) is equivalent to the following condition.

$$\theta H \neq H.$$

On the other hand, by Kobayashi's criterion (Fact 2.2) for proper actions, the condition (i) is equivalent to $\mathbf{R}H \pitchfork \mathfrak{a}(K_{\mathbf{C}})$, This amounts to

$$\mathbf{R}H \cap W\mathfrak{a}^\theta = \{0\}$$

by Lemma 4.3. Hence, to see the equivalence (i) $\Longleftrightarrow$ (ii), it is sufficient to prove

$$\theta H \neq H \Leftrightarrow \mathbf{R}H \cap W\mathfrak{a}^\theta = \{0\}.$$

First, suppose $\theta H = H$. Then $H \in \mathfrak{a}^\theta$. Therefore, $\mathbf{R}H \cap W\mathfrak{a}^\theta = \mathbf{R}H \neq \{0\}$.

Next, suppose $\theta H \neq H$. This happens only when $\mathfrak{g}_{\mathbf{C}} \simeq \mathfrak{so}(4n, \mathbf{C})$ by Theorem 4.1. Moreover, θ must be an outer automorphism. Hence, $\mathfrak{k}_{\mathbf{C}}$ must be isomorphic to $\mathfrak{so}(p,\mathbf{C}) + \mathfrak{so}(q,\mathbf{C})$ such that pq is odd and $p + q = 4n$. In this case, we shall see

$$\left(\bigcup_{w\in W} w\mathfrak{a}^\theta\right) \cap \mathfrak{a}_+ = \mathfrak{a}^\theta \cap \mathfrak{a}_+$$

in Lemma 5.2 below, and consequently,

$$\mathbf{R}H \cap (W\mathfrak{a}^\theta \cap \mathfrak{a}_+) = \mathbf{R}H \cap (\mathfrak{a}^\theta \cap \mathfrak{a}_+) = \{0\}.$$

Since $H \in \mathfrak{a}_+$, we conclude $\mathbf{R}H \cap W\mathfrak{a}^\theta = \{0\}$. □

Let $(G_{\mathbf{C}}, K_{\mathbf{C}}) = (SO(p+q,\mathbf{C}), SO(p,\mathbf{C}) \times SO(q,\mathbf{C}))$, such that pq is odd. We set $p + q = 2n$. Then, $\mathfrak{a} \simeq \mathbf{R}^n$, $\mathfrak{a}^\theta$ is of codimension one in $\mathfrak{a}$, and the Weyl group W is isomorphic to $\mathcal{S}_n \ltimes (\mathbf{Z}/2\mathbf{Z})^{n-1}$.

Lemma 5.2.

$$\left(\bigcup_{w\in W} w\mathfrak{a}^\theta\right) \cap \mathfrak{a}_+ = \mathfrak{a}^\theta \cap \mathfrak{a}_+.$$

Proof. By taking the standard basis $e_1, \ldots, e_n$ such that

$$\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}) = \{e_i \pm e_j :\ 1 \le i < j \le n\},$$

we have a coodinate expression of $\mathfrak{a}_+$ and $\mathfrak{a}^\theta$ as

$$\mathfrak{a}_+ = \{(x_1, \ldots, x_n) \in \mathbf{R}^n :\ x_1 \ge \cdots \ge x_{n-1} \ge |x_n|\},$$
$$\mathfrak{a}^\theta = \{(x_1, \ldots, x_{n-1}, 0) \in \mathbf{R}^n :\ x_1, \ldots, x_{n-1} \in \mathbf{R}\}.$$

Hence, we have

$$\bigcup_{w\in W} w\mathfrak{a}^\theta = \bigcup_{j=1}^n \{(x_1, \ldots, x_n) \in \mathbf{R}^n :\ x_j = 0\}$$

Therefore,

$$\begin{aligned}\bigcup_{w\in W} w\mathfrak{a}^\theta \cap \mathfrak{a}_+ &= \bigcup_{j=1}^n \{(x_1, \ldots, x_{j-1}, 0, \ldots, 0) \in \mathbf{R}^n \\ &\qquad :\ x_1 \ge \cdots \ge x_{j-1} \ge 0\} \\ &= \mathfrak{a}^\theta \cap \mathfrak{a}_+.\end{aligned}$$ □

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