Relative versions of theorems of Bogomolov and Sukhanov over perfect fields

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Abstract: In this paper, we investigate some aspects of representation theory of reductive groups over non-algebraically closed fields. Namely, we state and prove relative versions of well-known theorems of Bogomolov and of Sukhanov, which are related to observable and quasi-parabolic subgroups of linear algebraic groups over non-algebraically closed perfect fields.

Key words: Instability; observable subgroups; quasi-parabolic subgroups.

Introduction. The well-known notion of observability for closed subgroups of linear algebraic groups plays an important role in algebraic and geometric invariant theory (see, e.g. [10,12,13]). It characterizes a property of closed subgroups of a given algebraic group G as stabilizer subgroups via representations of G. Recently, due to some need for arithmetical applications (see, e.g., [19]), relative versions of some basic theorems in general, and in particular, related to this notion, have been proved in [1,3,15,17,19]. In this note we establish another results, which are relative versions of important theorems due to Bogomolov and Sukhanov, which have close relation with instability theory of Kempf [11] and its refinements due to Ramanan and Ramanathan [14], (which have been further refined by Coiai and Holla [9]).

Throughout this paper, we will work only with linear algebraic groups and we use freely standard notation, notions and results from [4,5]. For a linear algebraic group G, we always denote G° the connected component of G, $R_u(G)$ the unipotent radical of G, $DG := [G,G]$ the derived subgroup of G, $GL(V)$ the general linear group (of automorphisms of the vector space V). We denote by $\langle H\rangle$ the subgroup generated by the subset H, $\mathbf{G}_m$ the multiplicative group of the affine line $\mathbf{A}^1$, GL_n the general linear group, PGL_n the corresponding projective linear group, $\mathbf{P}_n$ the projective space of dimension n, all of which are defined over the prime field contained in k. We will work mostly over an infinite perfect field k, with a fixed algebraic closure $\bar{k}$, though many results hold for arbitrary fields. By a representation of a linear algebraic group G we always understand a *linear* one, i.e., a morphism of algebraic groups $\rho : G \to GL(V)$ for some finite dimensional vector space V and V is called then a G-module. If, moreover, V is a finite dimensional k-vector space of dimension n, and G, ρ are defined over k, then we also write $\rho : G \to GL_n$.

1. Preliminaries and statements of main results. In this section, we recall some basic notions and facts about observable subgroups, and some related notions. For more details we refer the readers to [4,5,10,11,14,17].

1.1. Let G be a linear algebraic group and let V be an irreducible G^0-module. Then $R_u(G)$ acts trivially on V and V is actually an irreducible $G^0/R_u(G)$-module. Since $G^0/R_u(G)$ is reductive, if V is an irreducible G^0-module, then a vector $v \in V$ is called following [10, p. 42], a *highest weight vector* if v is highest weight vector by considering V as a $G^0/R_u(G)$-module.

1.2. a) A closed subgroup Q of G^0 is said to be *quasi-parabolic* if $Q = G_v$ for a highest weight vector for some irreducible G^0-module V (cf. [10, p. 42]).
b) A closed subgroup H of G is called *subparabolic* if there is a quasi-parabolic subgroup Q of G^0 such that $H^0 \subseteq Q$ and $R_u(H) \subseteq R_u(Q)$ (cf. [10, p. 42]). (We say H is subparabolic with respect to Q for short.)
a′) For a k-group G, a subgroup Q of G^0 is said to be *quasi-parabolic over* k (or quasi-parabolic k-subgroup) if Q is defined over k and Q is quasi-parabolic.
b′) For a k-group G, a subgroup H of G is called

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subparabolic over k if it is defined over k and there is a quasi-parabolic k-subgroup Q of G^0 such that $H^0 \subseteq Q$ and $R_u(H) \subseteq R_u(Q)$.

c′) For a k-group G, a subgroup Q of G^0 is said to be *k-quasi-parabolic* if $Q = G_v$ for a highest weight vector $v \in V(k)$ for some irreducible $k - G^0$-module V.

d′) For a k-group G, a subgroup H of G is called *k-subparabolic* if it is defined over k and there is a k-quasi-parabolic subgroup Q of G^0 such that $H^0 \subseteq Q$ and $R_u(H) \subseteq R_u(Q)$.

1.3. We recall now the notion of observable subgroups. A closed subgroup H of linear algebraic group G is called an *observable subgroup* if the homogeneous space G/H is a quasi-affine variety. There are some ways to characterize observable ($k-$)subgroups (see e.g. [10] and also [17]), and here we give only one. Namely,

1.3.1. Theorem (Cf. [10, p. 21]). *The following statements are equivalent.*

1) *H is observable subgroup of G*;

2) *There exists a representation $\rho: G \to GL(V)$, such that for some $v \in V$, $H = G_v$, the stabilizer group of v in G.*

One may define a relative notion of the observability, namely for a linear algebraic group G defined over a field k, and we have the following relative version of Grosshans Theorem.

1.3.2. Theorem (Cf. [17, p. 438]). *The following statements are equivalent.*

1′) *H is observable in G and H is defined over k*;

2′) *There exists a k-representation $\rho: G \to GL(V)$, such that for some $v \in V(k)$, $H = G_v$, the stabilizer group of v in G.*

If H satisfies one of these conditions, it is also *k-observable* (see [17] for more details). Besides some important characterizations of observable subgroups as recalled above, we have the following theorems, which are main results of this note. The first one is a relative version of an important result due to A. Sukhanov (proved for algebraically closed fields).

Theorem A (Compare [16], Theorem 1, [10], Theorem 7.3). *Let k be a perfect field and let G be a linear algebraic group defined over k. A closed k-subgroup H of G is a k-observable subgroup of G if and only if H is a k-subparabolic subgroup of G.*

In Section 4 we will sketch a proof of this theorem. In fact, this result is contained in stronger Theorem C (Section 4.6). The proof (as in the absolute case) makes an essential use of the following relative version of an important theorem from geometric invariant theory (due to F. Bogomolov in the case of algebraically closed fields), in the form given by [10].

Theorem B (Compare [7], Theorem 1, [10], Theorem 7.6). *Let k be a perfect field, G a connected reductive k-group and let V be a finite dimensional k-G-module. Let $v \in V(k) \setminus \{0\}$. If v is unstable for the action of G on V (i.e., $0 \in \overline{G.v}$), then G_v is contained in a proper k-quasi-parabolic subgroup Q of G.*

The details of the proofs will be published somewhere else.

2. Some results from representation theory. Let T be a maximal torus of a linear algebraic group G. We set $X^*(T) := Hom(T, \mathbf{G}_m)$, the character group of T and $X_*(T) := Hom(\mathbf{G}_m, T)$, the set of 1-parameter subgroups (1-PS for short) of T and denotes $\langle ., . \rangle$ the usual dual pairing between $X^*(T)$ and $X_*(T)$. Let $\Gamma(G)$ denote the set of all 1-parameter subgroups of G.

2.1. Any inner product $(.,.)$ (i.e., symmetric non-degenerate pairing) on $X^*(T)$ (resp. on $X_*(T)$) via the duality, defines another one $(.,.)$ on $X_*(T)$ (resp. $X^*(T)$). For $\lambda \in X_*(T)$ (resp. $\chi \in X^*(T)$) we denote by $\chi_\lambda \in X^*(T)$ (resp. $\lambda_\chi \in X_*(T)$) the dual of λ (resp. χ), for a given inner product, namely $\langle \chi_\lambda, \lambda' \rangle := (\lambda, \lambda')$ for all $\lambda' \in X_*(T)$, and $\langle \chi', \lambda_\chi \rangle = (\chi', \chi)$, for all $\chi' \in X^*(T)$.

2.2. We need the following well-known (see [4,5,10,11,14])

2.2.1. Proposition. *Assume that k is an perfect field, G a connected reductive k-group, and T is a maximal torus of G defined over k. Then there exists an inner product $(.,.)$ on $X_*(T) \otimes_{\mathbf{Z}} \mathbf{R}$ such that the following conditions are satisfied*:

a) *For all $\lambda, \mu \in X_*(T)$ then $(\lambda, \mu) \in \mathbf{Z}$*;

b) *For all $w \in W(T, G)$ (Weyl group), we have $({}^w\lambda, {}^w\mu) = (\lambda, \mu)$*;

c) *The inner product is defined over k, i.e., $({}^\sigma\lambda, {}^\sigma\mu) = (\lambda, \mu), \forall \sigma \in \Gamma := Gal(k_s/k)$.*

In the sequel, we fix once for all such an inner product.

2.3. Let T be a maximal torus of a connected reductive group G, S the connected center of G. Denote by $\Phi(T, G)$, or just Φ, the root system of G with respect to T. For a 1-PS λ of G contained in $X_*(T)$ we denote $P(\lambda) := \langle T, U_\alpha \mid \alpha \in \Phi(T,G), \langle \alpha, \lambda \rangle \geq 0 \rangle$, which is a parabolic subgroup of G (cf. e.g. [10,13]) called the *parabolic subgroup associated to* λ. We also define, for a character $\chi \in X^*(T)$, $P_\chi := \langle Ker\chi, U_\alpha \mid \alpha \in \Phi(T,G), (\alpha,\chi) \geq 0 \rangle$ and $P(\chi) := TP_\chi = \langle T, U_\alpha \mid \alpha \in \Phi(T,G), (\alpha,\chi) \geq 0 \rangle$. $P(\chi)$ is also a parabolic subgroup of G, called the *parabolic subgroup associated to* χ. It follows from the very definition, that we have $P(\lambda) = P(\chi_\lambda) = P(r\chi_\lambda)$, $P_\chi \subseteq P_{r\chi}$ and $R_u(P_\chi) = R_u(P_\chi) = R_u(P_{r\chi})$ for any χ and positive integer r. Also, it is well-known and easy to prove (see e.g. [7], Sec. 2.9), that $\chi := \chi_\lambda$ is a character on $P(\chi)$.

2.4. We need some fundamental theorems on representation theory of reductive groups over non-algebraically closed fields (cf. [5,18] for more details). We use the same notation as in [18].

2.5. Let G be a reductive group defined over a field k, and let T be a maximal k-torus of G. Denote by $\Phi(T,G)$, or just Φ, the root system of G with respect to T, by Δ a basis of Φ corresponding to a Borel subgroup B of G containing T. Let Φ^+ be the set of positive roots of Φ with respect to Δ. Let $T_s := T \cap DG$, $\Lambda := X^*(T)$, Λ_r be the subgroup generated by roots $\alpha \in \Phi(T,G)$, $\Lambda_0 := \langle \Lambda_r, \chi : \chi|_{T_s} = 1 \rangle$, the subgroup generated by Λ_r and those χ, which have trivial restriction to T_s. Let B be a Borel subgroup of G containing T, Λ_+ the subset of dominant weights (with respect to B) of Λ. For $\Gamma := Gal(k_s/k)$, $\gamma \in \Gamma$, $\chi \in \Lambda$, denotes the usual Galois action by ${}^\gamma\chi$, and one defines (after [5], Section 6 or [18], Section 3) the action of Γ on Λ as follows: $\gamma(\chi) := w({}^\gamma\chi)$, where w is the unique element from the Weyl group $W(T,G)$ such that $w({}^\gamma\Lambda_+) = \Lambda_+$. We denote by $(\Lambda_+)^\Gamma$ the set of Γ-invariant elements of Λ_+ with respect to the just defined action. Especially, if P is a parabolic k-subgroup of G, containing B, then for any $\chi \in X^*(P)$, we have (see [5], Section 6, p. 105) $\gamma(\chi) = {}^\gamma\chi$.

2.6. Let k be a field, D a finite dimensional k-algebra, and let X be a D-module. We denote by ${}_kGL_{X,D}$ the group functor which associates to each k-algebra A the group of $D \otimes_k A$-automorphisms of $X \otimes_k A$. In particular, if X is free D-module D^m, instead of ${}_kGL_{X,D}$ we just write ${}_kGL_{m,D}$ (or just $GL_{m,D}$, if k is clearly indicated from the text), and if $D = k$, we just write ${}_kGL_m$ (or just GL_m). A *D-G-representation* (or just D-representation) of a k-group G is just a k-homomorphism $G \to {}_kGL_{X,D}$ for some X as above. There are obvious notions of D-equivalent representations of G.

2.7. We need the following important results of Tits, which extend some known results for semisimple groups to reductive ones.

2.7.1. Theorem (Cf. [18], Lemme 3.2, Théorème 3.3)**.** *Let G be a reductive group defined over a field k. Keep the notation as above.*
1) *Let D be a central simple algebra over k. The restriction to DG of any absolutely irreducible D-representation with dominant weight λ gives rise to an absolutely irreducible D-representation with dominant weight $\lambda|_{T_s}$ of DG. Conversely, any absolutely irreducible D-representation of DG with dominant weight $\lambda|_{T_s}$ extends in a unique way to an absolutely irreducible D-representation of G with dominant weight λ.*
2) *Let $\lambda \in (\Lambda_+)^\Gamma$, the set of Γ-invariant elements. Then there exist a central division algebra D_λ over k, an absolutely irreducible D-representation $\rho_\lambda : G \to GL_{m,D_\lambda}$ with simple dominant weight λ. The algebra D_λ is unique up to isomorphism, and for a given D_λ, the representation ρ_λ is determined uniquely up to D_λ-equivalence. If $\lambda \in \Lambda_0$, or if G is quasi-split, then we have $D_\lambda = k$.*

In above notation, let k_λ be the fixed field of the stabilizer of λ in Γ, which is a finite separable extension of k. Using the restriction operation *rest* as in [18], we set

$$ {}^k\rho_\lambda := rest_{k_\lambda/k}(rest_{D_\lambda/k} \circ \rho_\lambda). $$

2.7.2. Theorem (Cf. [18], Théorème 7.2, iii)**.** *Let λ and λ' be dominant weights. The representations ${}^k\rho_\lambda$ and ${}^k\rho_{\lambda'}$ are equivalent if and only if there exists $\gamma \in \Gamma$ such that $\gamma(\lambda) = \lambda'$.*
We need in the sequel the following important characterization of stabilizers of highest weight vectors.

2.7.3. Proposition (Cf. [10], Corollary 3.6)**.** *Let notation be as above, and let $\chi \in X^*(T)$. Then P_χ is the stabilizer of a highest weight vector $w \in W$ for some irreducible G-module W with highest weight χ. Conversely, the stabilizer of any highest weight vector* (*with respect to a given Borel subgroup B of G*) *is of the form P_χ, where $\chi \in X^*(T)$ is a dominant character* (*with respect to B*).
We need a relative version of the above proposition in the sequel. We claim that the following relative version of Proposition 2.7.3 holds.

2.7.4. Proposition. *Let G be a reductive* *group defined over a perfect field k, T a maximal k-torus of G containing a maximal k-split torus of G, $\chi \in X^*(T)_k$. Then there is a positive integer r and an absolutely irreducible k-representation $G \to GL_n = GL(W)$ with highest weight $\chi' = r\chi$, such that $P_{\chi'}$ is the stabilizer of a highest weight vector $w \in W(k)$. Conversely, for any absolutely irreducible k-representation $G \to GL_n = GL(W)$, the stabilizer of any highest weight vector $w \in W(k)$ (with respect to a given Borel subgroup B of G) is of the form P_χ, where $\chi \in X^*(T)_k$ is a dominant character (with respect to B).*

2.8. Instability theory of Kempf. Let k be a perfect field, G a connected reductive k-group, with a k-representation $\rho : G \to GL(V)$, $T \subset G$ a maximal k-torus. Let $(.,.)$ an inner product on $X_*(T) \otimes_{\mathbf{Z}} \mathbf{R}$ defined over k as in Proposition 2.2.1, and let $\|.\|$ denote the corresponding norm. For $\chi \in X^*(T)$ let $V_\chi := \{v \in V | \rho(t).v = \chi(t)v, \forall t \in T\}$ be the weight subspace of V corresponding to χ then $V = \oplus_\chi V_\chi$, and for any $v_0 \in V$, $v_0 = \oplus v_\chi$. Define the *state* of v_0 by $S_T(v_0) := \{\chi | v_\chi \neq 0\}$, and $\mu(v_0, \lambda) := Inf\{(\chi, \lambda) | \chi \in S_T(v_0)\}$, for $\lambda \in X_*(T)$, and set $V_i := \{v \in V | \lambda(s)v = s^i v, \forall s \in \mathbf{G}_m\}$, $i \in \mathbf{Z}$. Then one has $V_i = \oplus_{\langle \chi, \lambda \rangle = i} V_\chi$. We need the following basic result due to G. Kempf.

2.8.1. Theorem (Cf. [11], Theorem 4.2)**.** *With notation as above, let $v_0 \in V(k)$ be a non-zero unstable vector for the action of G.*
a) *The function $\nu(v_0, .) : \lambda \mapsto \mu(v_0, \lambda)/\|\lambda\|, \lambda \in \Gamma(G)$ attains a maximum value B_{v_0} on $\Gamma(G)$.*
b) *The set Λ_{v_0} of indivisible 1-PS's $\lambda \in \Gamma(G)$ defined over k such that $\nu(v_0, \lambda) = B_{v_0}$ is non-empty, and the k-parabolic subgroups $P(\lambda)$ of G are the same for all $\lambda \in \Lambda_{v_0}$, which is denoted by $P(v_0, \lambda)$.*
c) *For each maximal k-torus T of $P(v_0, \lambda)$, there exists a unique instability 1-PS $\lambda \in \Lambda_{v_0}$ such that $Im(\lambda) \subseteq T$, which is called the instability 1-PS for v_0 of T.*

2.8.2. With notation as above it is easy to see that if k is a perfect field and $\lambda \in X_*(T)_k$ then $\chi_\lambda \in X^*(T)_k$ (hence also $\chi_\lambda \in X^*(P(\lambda))_k$). From [14], Section 1.8, p.274, we know that for each $\lambda \in X_*(G)$, the vector space $V^j = \oplus_{i \geq j} V_i$ is stable under the action of $P(\lambda)$ through representation ρ, so we have a natural action of $P(\lambda)$ on V^j/V^{j+1}. Then we have the following important result.

2.8.3. Theorem ([14], Prop. 1.12, p. 276, [10], pp. 44–45)**.** *Assume λ is the instability one-parameter subgroup for v_0 and let $j = \mu(v_0, \lambda)$. Then there exists a positive integer d and a non-constant homogeneous function f on V^j/V^{j+1} such that $f(\pi(v_0)) \neq 0$ and $f(p.\pi(v)) = (\chi_\lambda)^d(p) f(\pi(v))$ for all $v \in V^j$, $p \in P(\lambda)$ and $\pi : V^j \to V^j/V^{j+1}$ is the natural projection.*

With notation as above we have

2.8.4. Corollary. *Let $\rho : G \to GL(V)$ be a representation, v a non-zero unstable vector in V, λ an 1-PS of v as in Theorem 2.8.1. Then for some positive integer d as in Theorem 2.8.3, $G_v \subseteq Ker(d.\chi_\lambda)$.*

3. A relative version of a theorem of Bogomolov. The main result of this section is the following relative version of an important theorem of Bogomolov. As an application, it will be used in the proof of a relative version of Sukhanov' Theorem, which is very close to it in describing the nature of stabilizers.

3.1. Theorem. *Let G be a reductive group defined over a perfect field k and let $\rho : G \to GL(V)$ be a k-rational representation. Assume that $v \in V(k) \setminus \{0\}$ is an unstable vector of representation ρ (i.e $0 \in \overline{G.v}$). Then there exists an absolutely irreducible representation $\pi : G \to GL(W)$ defined over k and a highest weight vector $w \in W(k)$ such that $G_v \subseteq G_w$, where the latter is a proper subgroup of G.*

We give the ideas of two proofs of Theorem 3.1.

3.2. *First proof* of this theorem is based on the proof of the absolute case of Theorem 3.1 as it was given in [10], Section 7, which makes use of main results of Kempf theory [11] (as stated above), with refinements due to Ramanan–Ramanathan [14]. Also, we make use of results of Chevalley–Tits theory on representations of reductive groups over non-algebraically closed fields presented above (Sections 2.3–2.8), and also the following

3.3. Lemma. *For a dominant weight $\chi \in \Lambda_+$ with respect to the Borel subgroup B containing T, assume that there exists a character $\tilde{\chi} \in X(P(\chi))$ such that $\tilde{\chi}|_T = \chi$. Let $\rho : G \to GL(W)$ be the irreducible representation corresponding to dominant weight χ and let $w \in W$ be a highest weight vector with weight χ. Then $Ker\, \tilde{\chi} = G_w$.*

3.4. The following consequence of the proof shows that if H is a quasi-parabolic k-subgroup, then it is also and k-subparabolic. A stronger assertion will be given in Theorem 4.1.

3.5. Proposition. *Let G be a reductive group defined over a perfect field k, T a maximal k-torus* *containing a maximal k-split torus of G. Fix a Borel subgroup containing T, which in turn, is contained in a minimal parabolic k-subgroup of G. Let $\rho : G \to GL(V) \simeq_k GL_n$ be an irreducible $\bar{k}$-representation with dominant weight $l_\rho = \chi \in X^*(T)_k$. If $v \in V(\chi)$ is a highest weight vector, such that its stabilizer $H := G_v$ is a proper subgroup defined over k, then H is k-subparabolic in a proper k-quasi-parabolic subgroup.*

3.6. Second proof of Theorem 3.2. This proof is based on some arguments given in [5], Section 12. First we need the following

3.6.1. Lemma (Cf. [5], p. 138)**.** *Assume that G is a connected semisimple group, H a connected reductive group, and K is connected semisimple, $\pi : H \to K$ is a surjective homomorphism between two algebraic groups, which induces a central isogeny from DH onto K, and that $\rho_1, \rho_2 : G \to H$ are two homomorphisms such that $\pi \circ \rho_1 = \pi \circ \rho_2$. Then we have $\rho_1 = \rho_2$.*

3.6.2. Corollary. *Suppose that G is a connected semi-simple group defined over a perfect field k, $\pi : GL_n \to PGL_n$ the projection, $\rho : G \to GL_n(\bar{k})$ is a $\bar{k}$-representation such that the induced projective representation $\pi \circ \rho : G \to PGL_n(\bar{k})$ is defined over k. Then ρ is defined over k.*

We apply this lemma to prove that

3.6.3. Lemma. *Assume that G is a connected reductive group defined over a perfect field k, T is a maximal k-torus of G, B is a Borel subgroup containning T, and $\chi \in X^*(T)_k$ is a dominant weight. Let $\pi : G \longrightarrow GL(V) \cong GL_n$ be an absolutely irreducible $\bar{k}$-representation corresponding to χ. Suppose that there exists a vector $v_0 \in V(k)(= k^n)$ of highest weight χ and that the induced projective representation $\overline{\pi} : G \to PGL_n(\bar{k})$ is defined over k. Then π is defined over k.*

The rest of the proof follows closely the line given in [5], by making use of Chatelet theorem on the isomorphism of Severi-Brauer variety with a rational point to a projective space $\mathbf{P}_n$ (see [8]), and Tits' Theorem 2.7.2.

4. A relative version of a theorem of Sukhanov. In this section, we assume that k is a perfect field. The purpose of this section is to give the following application of relative Bogomolov's Theorem (Section 3), namely the following relative version of a theorem of Sukhanov.

4.1. Theorem. *Suppose that k is a perfect field, G a linear algebraic group and $H < G$ is a subgroup of G, both are defined over k. Then the following statements are equivalent.*
a) H is an observable k-subgroup of G.
b) H is a k-subparabolic subgroup over k of G, i.e. it is defined over k and there exists a k-quasi-parabolic k-subgroup Q of G such that $H^0 < Q$ and $R_u(H) < R_u(Q)$.

4.2. The proof is based on the original proof (over algebraically closed fields) (see [16] or [10], Sections 2–4 (with some refinements)), which will be suitably adapted, and by making use of some results of previous sections and also main results of of [17]. In order to prove this theorem, we need to use the relative version of Bogomolov's Theorem 3.1 above, and we need also the following preparation results.

4.3. Theorem. *a*) (Cf. [10], Corollary 2.2., [17], Proposition 5.) *With above notation, H is $(k-)$observable in G if and only if H° is $(k-)$observable in G°.*
b) ([10], Corollary 2.10.) *Let H be a closed subgroup of G, normalized by a maximal torus T of G. Assume that L is an observable subgroup of G, such that $R_u(H) < R_u(L)$. Then H and TR_uL are observable in G.*
c) ([10], Corollary 2.3.) *Let $K < L < G$, such that K is observable in L, and L is observable in G. Then K is observable in G.*
d) ([10], Corollary 2.11.) *Let H be an observable subgroup of G. Then $H.R_u(G)$ is also observable subgroup in G.*
e) ([10], Theorem 7.1, [2]) *Let L be a linear algebraic group and let H be a closed subgroup of L such that $R_u(H) < R_u(L)$. Then L/H is affine.*

Also, we need the following relative version of [10], Theorem 3.9.

4.4. Theorem (Compare [10], Theorem 3.9.)**.** *Let G be a connected reductive group defined over a perfect field k, T a maximal k-torus of G, and let H be a closed k-subgroup of G which is normalized by T. Then H is observable in G if and only if for some $\chi \in X^*(T)_k$, H is a k-subparabolic subgroup in the k-quasi-parabolic subgroup P_χ of G.*

4.5. Next we need the following two lemmas, which cover some partial cases of Theorem 4.1.

4.5.1. Lemma (Cf. [10], Lemma 7.7.)**.** *Let G be a connected reductive k-group, and let H be a non-reductive connected observable k-subgroup of G. Then H is contained in a proper k-quasi-parabolic subgroup Q of G.*

4.5.2. Lemma (Cf. [10], Lemma 7.8.). *If G is a connected, reductive k-group and H is a connected observable k-subgroup of G, then H is k-subparabolic in G.*

The proof of Theorem 4.1 follows from these results in combination with Theorem 3.1.

4.6. Now we are able to derive the following stronger result about logical scheme of relations between quasi-parabolic and subparabolic subgroups over perfect fields.

Theorem C. *Let G be a linear algebraic group defined over a perfect field k, H a k-subgroup of G. We have the following implications.*

$$\begin{aligned} H \text{ is } k\text{-quasiparabolic} &\overset{1}{\Rightarrow} H \text{ is quasiparabolic over } k \\ &\overset{2}{\Rightarrow} H \text{ is observable over } k \\ &\overset{3}{\Leftrightarrow} H \text{ is } k\text{-subparabolic} \\ &\overset{4}{\Leftrightarrow} H \text{ is subparabolic over } k. \end{aligned}$$

If, moreover, G has positive semisimple rank and H is quasi-parabolic with respect to a separable character, then H is also k-quasi-parabolic.

The idea of the proof is as follows. The first and the second implication follow from the definition; the third equivalence relation follows from Theorem 4.1. For the last equivalence relation, $\Rightarrow$ follows from definition. For the converse ($\Leftarrow$), if H is subparabolic over k, then it is also $\bar{k}$-subparabolic, i.e, observable over $\bar{k}$ by Theorem 4.1, hence also observable over k, by [17], Theorem 9. The last statement follows by applying methods of proof of Theorem 3.1, in combination with Proposition 2.7.4.

Some other relations between these properties and related questions will be the subject of a paper under preparation.

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References

[1] A. Asok, B. Doran and F. Kirwan, Yang-Mills theory and Tamagawa numbers: The fascination of unexpected links in mathematics, Bull. London Math. Soc. 2008.

[2] A. Bialinycki-Birula, On homogeneous affine spaces of linear algebraic groups, Amer. J. Math. **85** (1963), 577–582.

[3] F. Bien and A. Borel, Sous-groupes épimorphiques de groupes linéaires algébriques. I, C. R. Acad. Sci. Paris Sér. I Math. **315** (1992), no. 6, 649–653.

[4] A. Borel, *Linear algebraic groups*, Second edition, Springer, New York, 1991.

[5] A. Borel and J. Tits, Groupes réductifs, Inst. Hautes Études Sci. Publ. Math. No. 27 (1965), 55–150.

[6] A. Borel and J. Tits, Compléments à l'article: "Groupes réductifs", Inst. Hautes Études Sci. Publ. Math. No. 41 (1972), 253–276.

[7] F. A. Bogomolov, Holomorphic tensors and vector bundles on projective varieties, Math. U. S. S. R. Izvestiya **13** (1979), 499–555.

[8] F. Châtelet, Variations sur un thème de H. Poincaré, Ann. Sci. École Norm. Sup. (3) **61** (1944), 249–300.

[9] F. Coiai and Y. Holla, Extension of structure group of principal bundle in positive characteristic, J. Reine Angew. Math. **595** (2006), 1–24.

[10] F. D. Grosshans, *Algebraic Homogeneous Spaces and Invariant Theory.* Lec. Notes in Math. 1673. Springer, Verlag, 1997.

[11] G. R. Kempf, Instability in invariant theory, Ann. of Math. (2) **108** (1978), no. 2, 299–316.

[12] S. Mukai, *An introduction to invariants and moduli*, Translated from the 1998 and 2000 Japanese editions by W. M. Oxbury, Cambridge Univ. Press, Cambridge, 2003.

[13] D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, Third edition, Springer, Berlin, 1994.

[14] S. Ramanan and A. Ramanathan, Some remarks on instability flag, Tohoku Math. J. (2) **36** (1984), no. 2, 269–291.

[15] C. S. Seshadri, Geometric reductivity over arbitrary base, Advances in Math. **26** (1977), no. 3, 225–274.

[16] A. A. Sukhanov, Description of the observable subgroups of linear algebraic groups, Mat. Sb. (N.S.) **137(179)** (1988), no. 1, 90–102, 144; translation in Math. USSR-Sb. **65** (1990), no. 1, 97–108.

[17] N. Q. Thǎńg and D. P. Bǎć, Some rationality properties of observable groups and related questions, Illinois J. Math. **49** (2005), no. 2, 431–444.

[18] J. Tits, Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque, J. Reine Angew. Math. **247** (1971), 196–220.

[19] B. Weiss, Finite dimensional representations and subgroup actions on homogeneous spaces, Israel J. Math. **106** (1998), 189–207.