Ruelle zeta function for odd dimensional hyperbolic manifolds with cusps

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(Communicated by Shigefumi MORI, M.J.A., Dec. 12, 2007)

Abstract: In this paper we announce fundamental results of the Ruelle zeta function for odd dimensional hyperbolic manifolds with cusps; the meromorphic extension over $\mathbf{C}$, its functional equation and the singularity at $s = 0$.

Key words: Ruelle zeta function; Selberg zeta function.

1. Introduction. Let us consider an odd dimensional noncompact hyperbolic manifold

$$X_\Gamma = \Gamma\backslash \mathrm{SO}_0(2n+1,1)/\mathrm{SO}(2n+1)$$

where Γ is a cofinite discrete subgroup of $G = \mathrm{SO}_0(2n+1,1)$ and $K = \mathrm{SO}(2n+1)$ is a maximal compact subgroup of $\mathrm{SO}_0(2n+1,1)$. Throughout this paper, we assume that the group generated by the eigenvalues of Γ contains no root of unity. Its consequences are that Γ is torsion free and

$$\Gamma \cap P = \Gamma \cap N$$

for a Γ-cuspidal minimal parabolic subgroup P and a Langlands decomposition $P = MAN$ where $M = \mathrm{SO}(2n) \subset K = \mathrm{SO}(2n+1)$. Then X_Γ has a negative constant curvature with respect to the metric induced from the Killing form over the Lie algebra of G.

Let ρ be a finite-dimensional unitary representation of $\pi_1(X_\Gamma) = \Gamma$. For such a manifold X_Γ and ρ, the Ruelle zeta function $R_\rho(s)$ is defined by

$$R_\rho(s) := \prod_\gamma \det\bigl(\mathrm{Id} - \rho(\gamma)e^{-s\,l_\gamma}\bigr)^{-1}$$

for $\Re(s) > 2n$. Here the product is given over the Γ-conjugacy classes of the primitive hyperbolic element γ in Γ, the determinant denoted by det is taken over the representation space V_ρ of ρ, and l_γ denotes the length of the prime geodesic determined by γ. The fundamental questions for $R_\rho(s)$ are its meromorphic extension over $\mathbf{C}$ and functional equation. These questions could be answered by proving the corresponding properties of the Selberg zeta function $Z_\rho(\sigma, s)$ where $Z_\rho(\sigma, s)$ is attached to a representation σ of $M = \mathrm{SO}(2n)$. For this, we need to analyze all the terms in Selberg trace formula and such a analysis has been available for compact hyperbolic manifold X_Γ. Hence the above questions for $R_\rho(s)$ can be easily answered and solutions have been well known now for compact X_Γ. For instance, see [3] for even dimensional case which is more difficult case than odd dimensional case. In this paper, we give the answers to these questions for odd dimensional hyperbolic manifold with cusps. For this, we also obtain the meromorphic extension of $Z_\rho(\sigma, s)$ over $\mathbf{C}$ with precise information about its zeros and poles and the functional equation of $Z_\rho(\sigma, s)$ for odd dimensional hyperbolic manifold with cusps.

2. Results for $Z_\rho(\sigma, s)$. Let us recall the definition of the Selberg zeta function $Z_\rho(\sigma, s)$,

$$Z_\rho(\sigma, s) := \exp\left(-\sum_\gamma \frac{\chi_\rho(\gamma)\,\chi_\sigma(m_\gamma)}{j_\gamma\, D(\gamma)}\, e^{-(s-n)l_\gamma} \right)$$

defined for $\Re(s) > 2n$. Here the sum is given over the Γ-conjugacy classes of the hyperbolic element γ in Γ, l_γ denotes the length of the closed geodesic determined by γ and j_γ is the positive integer such that $\gamma = \gamma_0^{j_\gamma}$ for a primitive hyperbolic element γ_0,

$$D(\gamma) = e^{n\,l_\gamma}\Bigl|\det\Bigl(\mathrm{Ad}(m_\gamma a_\gamma)^{-1} - \mathrm{Id}\bigr|_{\mathfrak{n}}\Bigr)\Bigr|$$

for the element $m_\gamma a_\gamma \in MA^+$ which is conjugate to γ and χ_ρ, χ_σ denote the characters of ρ, σ respectively. When $M = \mathrm{SO}(2n)$, the representation ring of M is generated by the fundamental representation σ_k acting on $\wedge^k \mathbf{R}^{2n} \otimes \mathbf{C}$ for $k = 0, 1, \dots, (n-1)$ and the half spin representations σ_+, σ_- acting on $\wedge^n \mathbf{R}^{2n} \otimes \mathbf{C}$. We denote by $d(\sigma_k)$ the dimension of representation space of σ_k, $\wedge^k \mathbf{R}^{2n} \otimes \mathbf{C}$ for $0 \le k \le n-1$ and by $d(\sigma_n)$ the corresponding one of $\sigma_\pm$.

2000 Mathematics Subject Classification. Primary 11M36, 11F72.

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To state our results of $Z_\rho(\sigma, s)$ for $\sigma = \sigma_k$ and $\sigma_+ \oplus \sigma_-$, we need to introduce some notations. First let τ_k be the fundamental representation of $K = \mathrm{SO}(2n+1)$ acting on $V_{\tau_k} = \wedge^k \mathbf{R}^{2n+1} \otimes \mathbf{C}$ and Δ_k denote the Laplacian acting on the space of the smooth sections of the locally homogeneous vector bundle $E_{\rho,\tau_k} = V_\rho \times_\rho G \times_{\tau_k} V_{\tau_k}$. Since X_Γ is noncompact, we have the following decomposition

$$L^2\big(X_\Gamma, E_{\rho,\tau_k}\big) = L^2_d\big(X_\Gamma, E_{\rho,\tau_k}\big) \oplus L^2_c\big(X_\Gamma, E_{\rho,\tau_k}\big).$$

Now the actions of $\pi_{\sigma_k,i\lambda}(\Delta_k)$ and $\pi_{\sigma_{k-1},i\lambda}(\Delta_k)$ on $L^2_d\big(X_\Gamma, E_{\rho,\tau_k}\big)$ have the discrete eigenvalues of the forms $\lambda_j(k)^2 + (n-k)^2$ and $\lambda_j(k-1)^2 + (n-k+1)^2$ respectively. This is also true if (σ_k, k) is replaced by $(\sigma_\pm, n)$. Here $\pi_{\sigma_k,i\lambda} = \mathrm{Ind}_{MAN}^G(\sigma_k \otimes e^{i\lambda+\rho} \otimes 1_N)$ is a non-unitary principal series representation of G for $\sigma_k \in \hat{M}$ and $\lambda \in \mathbf{C} \simeq \mathrm{Lie}(A)^* \otimes \mathbf{C}$. If $\lambda \in \mathbf{R}$ then $\pi_{\sigma_k,i\lambda}$ is called a unitary principal series. Besides if $i\lambda \in [-(n-k), n-k]$ and $\pi_{\sigma_k,i\lambda}$ is unitarizable then called a complementary series representation. The spectral resolution of Δ_k over $L^2_c\big(X_\Gamma, E_{\rho,\tau_k}\big)$ is determined by the scattering operators $C^k_\rho(\sigma_k, s)$ and $C^k_\rho(\sigma_{k-1}, s)$. These have the matrix forms of size $d_c(\rho)$ where

$$d_c(\rho) = \sum_{j=1}^{\kappa} d_j(\rho).$$

Here κ denotes the number of cusps and $d_j(\rho)$ denotes the dimension of the maximal subspace of V_ρ over which $\rho|_{P_j\cap\Gamma}$ acts trivially where $P_1, \dots, P_\kappa$ denote representatives of Γ-conjugacy classes of Γ-cuspidal parabolic subgroups. Now for $\sigma = \sigma_\pm$, the scattering matrix has the size $2\,d_c(\rho)$ since $\sigma_\pm$ is un-ramified and we denote this by $C^n_\rho(\sigma_n, s)$. It is well known that $C^k_\rho(\sigma_k, s)$ has the meromorphic extension over $\mathbf{C}$ with poles for $\Re(s) < 0$ and finitely many real poles in the interval $(0, n]$ and satisfies the functional equation

$$C^k_\rho(\sigma_k, s) C^k_\rho(\sigma_k, -s) = \mathrm{Id}.$$

Now we can state our first result on the Selberg zeta function.

Theorem 2.1. *The Selberg zeta function $Z_\rho(\sigma_k, s)$ for $0 \le k \le n-1$ and $Z_\rho(\sigma_+, s) Z_\rho(\sigma_-, s)$ a priori defined for $\Re(s) > 2n$ have the meromorphic extensions over $\mathbf{C}$ with zeros at*

- *$s = n \pm i\lambda_j(k)$ of order $m_j(k)$ (of order $2m_j(k)$ if $\lambda_j(k) = 0$), where $\lambda_j(k)^2 + (n-k)^2$ is an eigenvalue of $\pi_{\sigma_k,i\lambda}(\Delta_k)$ with multiplicity $m_j(k)$,*
- *$s = n + q_j$ of order $d(\sigma_k) b_j$ where $\det C^k_\rho(\sigma_k, s)$ has a pole at $s = q_j$ of order b_j with $\Re(q_j) < 0$,*

and poles at

- *$s = k$ of order $d_c(\rho)\, e(n,k)$ where*
$$e(n,k) := (-1)^{k+1} \left(\sum_{j=k+1}^{n} (-1)^j d(\sigma_j) \right) = \begin{pmatrix} 2n-1 \\ k \end{pmatrix} > 0$$
for $0 \le k < n$ and $e(n,n) := 0$,
- *$s = n$ of order $\frac{1}{2} d(\sigma_k) \mathrm{tr}\big(n(\sigma_k)\mathrm{Id} - C^k_\rho(\sigma_k, 0)\big)$, where $n(\sigma_k) = 1$ for $0 \le k < n$ and $n(\sigma_n) = 0$.*
- *$s = n - q_j$ of order $d(\sigma_k) b_j$ where $\det C^k_\rho(\sigma_k, s)$ has a pole at $s = q_j$ of order b_j with $0 < q_j \le n$,*
- *$s = n - \ell$ of order $d_c(\rho)\, d(\sigma_k)$ for $\ell \in \mathbf{N} - \{n-k\}$.*

The proof of Theorem 2.1 follows from the invariant form of the Selberg trace formula where the non-invariant two terms combined into one invariant term,

$$S_\rho(\sigma_k, s) = C^k_\rho(\sigma_k, s) C_{\tau_k}(\sigma_k, s)^{-1}$$

where $C_{\tau_k}(\sigma_k, s)$ denotes the Harish-Chandra C-function attached to (τ_k, σ_k). The zeros and poles of $Z_\rho(\sigma_k, s)$ can be stated in terms with $S_\rho(\sigma_k, s)$ instead of $C^k_\rho(\sigma_k, s)$.

The second result on the Selberg zeta function is its functional equation stated in the following theorem.

Theorem 2.2. *For $s \in \mathbf{C}$, the following equalities hold for $0 \le k \le n-1$,*

$$\begin{aligned} &Z_\rho(\sigma_k, s)\, \Gamma(s-n+1)^{-d_c(\rho) d(\sigma_k)} \\ &\quad \cdot (s-k)^{-d_c(\rho)\, d(n,k)} \det C^k_\rho(\sigma_k, 0)^{d(\sigma_k)} \\ &\quad \cdot \exp\left(\int_{n-k}^{s-k} 2P^n_{k,\rho}(i(z-n+k))\, dz \right) \\ &= Z_\rho(\sigma_k, 2n-s)\, \Gamma(n-s+1)^{-d_c(\rho)\, d(\sigma_k)} \\ &\quad \cdot (2n-k-s)^{-d_c(\rho)\, d(n,k)} \det C^k_\rho(\sigma_k, n-s)^{d(\sigma_k)} \end{aligned}$$

where $P^n_{k,\rho}(z)$ is an even polynomial of z and

$$d(n,k) := (-1)^k \left(\sum_{j=k}^{n} (-1)^j d(\sigma_j) \right) \ge 0,$$

and

$$\begin{aligned} &Z_\rho(\sigma_+, s) Z_\rho(\sigma_-, s) \Gamma(s-n+1)^{-2d_c(\rho)\, d(\sigma_n)} \\ &\quad \cdot \det C^n_\rho(\sigma_n, 0)^{d(\sigma_n)} \exp\left(\int_0^{s-n} 2P^n_{n,\rho}(iz)\, dz \right) \end{aligned}$$

$$= Z_\rho(\sigma_+, 2n-s) Z_\rho(\sigma_-, 2n-s) \cdot \Gamma(n-s+1)^{-2d_c(\rho)\, d(\sigma_n)} \ \det C^n_\rho(\sigma_n, n-s)^{d(\sigma_n)}$$

where $P^n_{n,\rho}(z)$ is an even polynomial of z.

Theorem 2.1 and 2.2 are generalizations of the results of Gangolli and Warner in [2] and Wakayama in [9] to the case of the nontrivial locally homogeneous vector bundle over noncompact hyperbolic manifold with cusps. The result for $Z_\rho(\sigma_+, s) Z_\rho(\sigma_-, s)$ with the trivial ρ has been also obtained in [7] where its relation with the determinant of the Dirac Laplacian is studied. The proofs of Theorem 2.1 and 2.2 are applications of (the invariant form of) the Selberg trace formula in [10] and an explicit computation of the weighted unipotent orbital integral analyzed in [5] to our cases. The details of proofs are given in [4].

3. Results for $\boldsymbol{R_\rho(s)}$. Let us recall the following equality which holds for $\Re(s) > 2n$,

$$R_\rho(s) = \prod_{k=0}^{2n} Z_\rho(\sigma_k, s+k)^{(-1)^{k+1}} \tag{1}$$

where $Z_\rho(\sigma_n, s+n) = Z_\rho(\sigma_+, s+n) Z_\rho(\sigma_-, s+n)$. Combining this equality and Theorem 2.1, we can easily obtain

Theorem 3.1. *The Ruelle zeta function $R_\rho(s)$ defined a priori for $\Re(s) > 2n$ has the meromorphic extension over* $\mathbf{C}$.

We can state explicitly poles and zeros of $R_\rho(s)$ using Theorem 2.1 and the equality (1). In particular, we can derive the order N_0 of the singularity of $R_\rho(s)$ at $s = 0$, that is, the integer such that $\lim_{s\to 0} s^{N_0} R_\rho(s)$ is a nonzero finite value.

Theorem 3.2. *The order N_0 of the singularity of $R_\rho(s)$ at $s = 0$ is*

$$2\sum_{k=0}^{n} (-1)^k (n+1-k)\beta_k + \sum_{k=0}^{n-1} (-1)^{k+1} d(\sigma_k) b_k + d_c(\rho) \sum_{k=1}^{n} (-1)^k \, k \, d(\sigma_k)$$

where $\beta_k := \dim\ker(\Delta_k)$ and b_k is the order of singularity of $\det C^k_\rho(\sigma_k, s)$ at $s = n-k$.

Theorem 3.2 is a generalization of Theorem 3 in [1,6] to the case of a noncompact hyperbolic manifold with cusps X_Γ where the second term (the scattering contribution) and the third term (the cuspidal contribution from the unipotent term) appear. It is also easy to derive the functional equation of $R_\rho(s)$ from Theorem 2.2.

Theorem 3.3. *The following functional equation of Ruelle zeta function $R_\rho(s)$ holds,*

$$R_\rho(-s) = R_\rho(s)\, Y(n,s)^{d_c(\rho)} \cdot \det C_\rho(n,s) \exp\bigl(-Q_\rho(s)\bigr)$$

where

$$Y(n,s) := Y_1(n,s)\, Y_2(n,s)$$

with

$$Y_1(n,s) := \prod_{k=0}^{n-1} \left(\frac{s+(n-k)}{s-(n-k)} \right)^{(-1)^k a(n,k)}$$

with

$$a(n,k) := 2e(n,k) - d(\sigma_k) = \frac{n-k}{n}\, d(\sigma_k)$$

for $0 \le k \le n-1$ and

$$Y_2(n,s) := \prod_{k=0}^{n-1} \left(\frac{s+2(n-k)}{s-2(n-k)} \right)^{(-1)^k d(n,k)},$$

$$C_\rho(n,s) := \prod_{k=0}^{n} \left(\widetilde{C}_\rho(\sigma_k, s) \right)^{(-1)^k d(\sigma_k)}$$

with

$$\widetilde{C}_\rho(\sigma_k, s) := C^k_\rho(\sigma_k, n-k-s)\, C^k_\rho(\sigma_k, -(n-k)-s)$$

for $0 \le k \le n-1$ and

$$\widetilde{C}_\rho(\sigma_n, s) := C^n_\rho(\sigma_n, -s) C^n_\rho(\sigma_n, 0)^{-1},$$

and

$$Q_\rho(s) := \sum_{k=0}^{n-1} (-1)^k \int_{-s}^{s} 2P^n_{k,\rho}(i(z-n+k))dz + (-1)^n \int_0^s 2P^n_{n,\rho}(iz)dz.$$

Applications of Theorem 3.2 and 3.3 to the relation with analytic torsion, which is a generalization of [1] to hyperbolic manifold with cusps, is given in [8].

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