Normal families and shared values of meromorphic functions

By Chunlin Lei, Mingliang Fang, and Degui Yang

Department of Applied Mathematics, South China Agricultural University, Guangzhou, 510642, P. R. China

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Abstract: Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let q, k be two positive integers, and let a, b be two non-zero complex numbers. If, for each $f \in \mathcal{F}$, the zeros of f have multiplicity at least $k+1$, and $f = a \Leftrightarrow G(f) = b$, where $G(f) = P(f^{(k)}) + H(f)$ be a differential polynomial of f satisfying $q \geq \gamma_H$, and $\frac{\Gamma}{\gamma}|_H < k+1$, then $\mathcal{F}$ is normal in D.

Key words: Normal families; meromorphic functions; shared values.

1. Introduction. Let f and g be meromorphic functions on a domain D, and let a and b be two complex numbers. If $g(z) = b$ whenever $f(z) = a$, we write

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If $f(z) = a \Leftrightarrow g(z) = a$, we say that f and g share a on D.

Let $a_i(z), (i = 1, 2, \ldots, q-1), b_j(z), (j = 1, 2, \ldots, n)$ be analytic in D, $n_0, n_1, \ldots, n_k$ be non-negative integers. Set

$$P(\omega) = \omega^q + a_{q-1}(z)\omega^{q-1} + \ldots + a_1(z)\omega,$$

$$M(f, f', \ldots, f^{(k)}) = f^{n_0}(f')^{n_1} \ldots (f^{(k)})^{n_k},$$

$$\gamma_M = n_0 + n_1 + \ldots + n_k,$$

$$\Gamma_M = n_0 + 2n_1 + \ldots + (k+1)n_k.$$

$M(f, f', \ldots, f^{(k)})$ is called the differential monomial of f, γ_M the degree of $M(f, f', \ldots, f^{(k)})$ and Γ_M the weight of $M(f, f', \ldots, f^{(k)})$.

Let $M_j(f, f', \ldots, f^{(k)}), (j = 1, 2, \ldots, n)$ be differential monomials of f. Set

$$H(f, f', \ldots, f^{(k)}) = b_1(z)M_1(f, f', \ldots, f^{(k)}) + \ldots + b_n(z)M_n(f, f', \ldots, f^{(k)}),$$

$$\gamma_H = \max\{\gamma_{M_1}, \gamma_{M_2}, \ldots, \gamma_{M_n}\},$$

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$H(f, f', \ldots, f^{(k)})$ is called the differential polynomial of f, γ_H the degree of $H(f, f', \ldots, f^{(k)})$ and Γ_H the weight of $H(f, f', \ldots, f^{(k)})$. Set

$$\frac{\Gamma}{\gamma}|_H = \max\{\frac{\Gamma_{M_1}}{\gamma_{M_1}}, \frac{\Gamma_{M_2}}{\gamma_{M_2}}, \ldots, \frac{\Gamma_{M_n}}{\gamma_{M_n}}\},$$

$$G(f) = P(f^{(k)}) + H(f, f', \ldots, f^{(k)}).$$

Schwick[1] was the first to draw a connection between values shared by functions in $\mathcal{F}$ and the normality of the family $\mathcal{F}$. Specifically, he proved the following theorem.

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and let a_1, a_2, a_3 be three distinct complex numbers. If, for each $f \in \mathcal{F}$, f and f' share a_1, a_2, a_3, then $\mathcal{F}$ is normal in D.*

Fang[2] proved the following theorem.

Theorem B. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let k be a positive integer, and let a be a non-zero complex number. If, for each $f \in \mathcal{F}$, $f \neq 0$, and $f = a \Leftrightarrow f^{(k)} = a$, then $\mathcal{F}$ is normal in D.*

Fang and Zalcman[3] improved Theorem B as follows:

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let k be a positive integer, and let a, b be two non-zero complex numbers. If, for each $f \in \mathcal{F}$, the zeros of f have multiplicity at least $k+1$, and $f = a \Leftrightarrow f^{(k)} = b$, then $\mathcal{F}$ is normal in D.*

In this paper, we extended Theorem C as follows:

Theorem 1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let q, k be two positive integers, and let a, b be two non-zero complex numbers. If, for each $f \in \mathcal{F}$, the zeros of f have multiplicity at least $k+1$, and $f = a \Leftrightarrow G(f) = b$, where $G(f) = P(f^{(k)}) + H(f)$ be a differential polynomial of f satisfying $q \geq \gamma_H$, and $\frac{\Gamma}{\gamma}|_H < k+1$, then $\mathcal{F}$ is*

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normal in D.

As an application of Theorem 1, we have the following example.

Example 1. Let k be a positive integer, let $f_n(z) = ne^z$, let $\mathcal{F}=\{f_n(z) : n = 1, 2, \ldots\}$, let $D = \{z : |z| < 1\}$, and let $G(f) = f^{(k)}$. Then $\mathcal{F}$ be a family of meromorphic functions in a domain D, for each $f \in \mathcal{F}$, $f \neq 0$ and $f = 1 \Leftrightarrow G(f) = 1$. By Theorem 1, we obtain that $\mathcal{F}$ is normal in D.

From Theorem 1, we can get

Corollary 2. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let $a_1(z), a_2(z), \ldots, a_k(z)$ be holomorphic functions in D, let k be a positive integer, and let a, b be two non-zero complex numbers. If, for each $f \in \mathcal{F}$, the zeros of f have multiplicity at least $k + 1$, and $f = a \Leftrightarrow L(f) = b$, where $L(f) = f^{(k)}+a_1(z)f^{(k-1)}+a_2(z)f^{(k-2)}+\ldots+a_k(z)f$, then $\mathcal{F}$ is normal in D.*

2. Some Lemmas. For the proof of Theorem 1, we need the following lemmas.

Lemma 1[4]**.** *Let k be a positive integer, let $\mathcal{F}$ be a family of functions meromorphic on the unit disc $\triangle$, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal at z_0, there exist, for each $0 \leq \alpha \leq k$,*

a) *points $z_n \in \triangle$, $z_n \to z_0$;*

b) *functions $f_n \in \mathcal{F}$; and*

c) *positive numbers $\rho_n \to 0$*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) = g_n(\zeta) \to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on C, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$. In particular, g has order at most 2.

Lemma 2[5]**.** *Let $f(z)$ be a meromorphic fuction of finite order in the plane, let k be a positive integer, and let b be a non-zero complex number. If the zeros of $f(z)$ have multiplicity at least $k+1$, the poles are multiple, and $f^{(k)}(z) \neq b$, then $f(z)$ is a constant.*

3. Proofs of Theorems 1. Without lose of generality we assume that $D = \{|z| < 1\}$. Suppose that $\mathcal{F}$ is not normal at point 0. Then by Lemma 1, for $\alpha = k$, there exist $f_j \in \mathcal{F}$, $z_j \to 0$, and $\rho_j \to 0^+$ such that $g_j(\zeta) = \rho_j^{-k} f_j(z_j + \rho_j\zeta)$ converges locally uniformly to a non-constant function $g(\zeta)$. Moreover, $g(\zeta)$ is of order at most 2 and only zeros of multiplicity at least $k + 1$. Set $Q(\omega) = \omega^q + a_{q-1}(0)\omega^{q-1} + \ldots + a_1(0)\omega$,

We claim that:

(i) $Q(g^{(k)}) \neq b$;

(ii) the poles of g are multiple .

Suppose now that $Q(g^{(k)}(\zeta_0)) = b$. we claim that $Q(g^{(k)}) \not\equiv b$. Otherwise, g must be a polynomial of exact degree k , which contradicts the fact that each zero of g has multiplicity at least k+1. Since $Q(g^{(k)})(\zeta_0) = b$. Obviously, $g(\zeta_0) \neq \infty$. Hence there exists $\delta > 0$ such that $g(\zeta)$ is analytic on $G_{2\delta} = \{\zeta : |\zeta-\zeta_0| < 2\delta\}$. Thus $g_j^{(i)}(\zeta)(i = 0, 1, 2, \ldots, k)$ are analytic on $G_\delta = \{\zeta : |\zeta - \zeta_0| < \delta\}$ for large j and $g_j^{(i)}(\zeta)$ converges uniformly to $g^{(i)}(\zeta)(i = 0, 1, 2, \ldots, k)$ on $\overline{G}_\delta = \{\zeta : |\zeta - \zeta_0| \leq \delta\}$.

As

$$\begin{aligned} G(f_j)(z_j + \rho_j\zeta) - b = {} & P(f_j^{(k)}(z_j + \rho_j\zeta)) \\ & +H(f_j, f_j', \ldots, f_j^{(k)})(z_j+\rho_j\zeta) \\ & -b, \end{aligned}$$

and

$$\begin{aligned} & H(f_j, f_j', \ldots, f_j^{(k)})(z_j + \rho_j\zeta) \\ & = \sum_{i=1}^{n} b_i(z_j + \rho_j\zeta)\rho_j^{(k+1)\gamma_{M_i}-\Gamma_{M_i}} \\ & \quad \times M_i(g_j, g_j', \ldots, g_j^{(k)})(\zeta). \end{aligned}$$

Considering $b_i(z)$ are analytic on D $(i = 1, 2, \ldots, n)$, we have

$$|b_i(z_j + \rho_j\zeta)| \leq M\left(\frac{1+r}{2}, b_i(z)\right) < \infty, \quad (i = 1, 2, \ldots, n)$$

for sufficiently large j.

Hence we deduce from $\frac{\Gamma}{\gamma}|_H < k + 1$ that

$$\sum_{i=1}^{n} b_i(z_j + \rho_j\zeta)\rho_j^{(k+1)\gamma_{M_i}-\Gamma_{M_i}} M_i(g_j, g_j', \ldots, g_j^{(k)})(\zeta)$$

converges uniformly to 0 on $D_{\frac{\delta}{2}} = \{\zeta : |\zeta - \zeta_0| < \frac{\delta}{2}\}$.

Thus we know that $G(f_j)(z_j+\rho_j\zeta)-b$ converges uniformly to $Q(g^{(k)}) - b$ on $D_{\frac{\delta}{2}} = \{\zeta : |\zeta - \zeta_0| < \frac{\delta}{2}\}$.

Hence, by Hurwitz's theorem we deduce that there exist ζ_j, $\zeta_j \to \zeta_0$ such that, for large j,

$$\begin{aligned} & P(g_j^{(k)}(\zeta_j)) + \sum_{i=1}^{n} b_i(z_j + \rho_j\zeta_j)\rho_j^{(k+1)\gamma_{M_i}-\Gamma_{M_i}} \\ & \quad \times M_i(g_j, g_j', \ldots, g_j^{(k)})(\zeta_j) = b. \end{aligned}$$

Thus

$$P(f_j^{(k)}(z_j+\rho_j\zeta_j))+H(f_j,f_j',\dots,f_j^{(k)})(z_j+\rho_j\zeta_j)=b.$$

It follows from $f=a \Leftrightarrow G(f)=b$ that

$$f_j(z_j+\rho_j\zeta_j)=a.$$

Thus

$$g_j(\zeta_j)=\frac{f_j(z_j+\rho_j\zeta_j)}{\rho_j^k}=\frac{a}{\rho_j^k}.$$

we have $g(\zeta_0)=\lim_{n\to\infty} g_j(\zeta_j)=\infty$, which contradicts $Q(g^{(k)}(\zeta_0))=b$. This proves (i) .

Now we prove (ii). Suppose $g(\zeta_0)=\infty$. Since $g\not\equiv\infty$, there exists a closed disc $K=\{\zeta:|\zeta-\zeta_0|\le\delta\}$ on which $1/g$ and $1/g_j$ are holomorphic (for j sufficiently large) and $1/g_j\to 1/g$ uniformly. Hence, $1/g_j(\zeta)-\rho_j^k/a\to 1/g(\zeta)$ on K, and since $1/g$ is nonconstant, there exist ζ_j, $\zeta_j\to\zeta_0$, such that (for j large enough)

$$\frac{1}{g_j(\zeta_j)}-\frac{\rho_j^k}{a}=0.$$

Hence $f_j(z_j+\rho_j\zeta_j)=a$. Thus we have

$$P(f_j^{(k)}(z_j+\rho_j\zeta_j))+H(f_j,f_j',\dots,f_j^{(k)})(z_j+\rho_j\zeta_j)=b.$$

Thus

$$\begin{aligned}(1)\qquad & P(g_j^{(k)}(\zeta_j))\\ &+\sum_{i=1}^{n} b_i(z_j+\rho_j\zeta_j)\rho_j^{(k+1)\gamma_{M_i}-\Gamma_{M_i}}\\ &\times M_i(g_j,g_j',\dots,g_j^{(k)})(\zeta_j)=b.\end{aligned}$$

We can get

$$(2)\qquad \left(\frac{1}{g_j}\right)'=-\frac{g_j'}{g_j^2},$$

$$(3)\qquad \left(\frac{1}{g_j}\right)''=-\frac{g_j''}{g_j^2}+2\frac{(g_j')^2}{g_j^3},$$

for $k\ge 3$, mathematical induction shows that

$$\left(\frac{1}{g_j}\right)^{(k)}=-\frac{g_j^{(k)}}{g_j^2}+k!\frac{(g_j')^k}{g_j^{k+1}}+\sum_{i=0}^{k-2}A_ig_j^i,$$

Thus

$$(4)\quad g_j^{(k)}=g_j^2\left[k!\frac{(g_j')^k}{g_j^{k+1}}+\sum_{i=0}^{k-2}A_ig_j^i-\left(\frac{1}{g_j}\right)^{(k)}\right].$$

Thus by (1), (2), (3), (4) and $q\ge\gamma_H$, we have

$$\begin{aligned}(5)\qquad & (k!)^q\left(\frac{g_j'(\zeta_j)}{g_j^2(\zeta_j)}\right)^{kq} g_j^{(k+1)q}(\zeta_j)\\ &+\sum_{i=0}^{(k+1)q-1} B_ig_j^i(\zeta_j)=b,\end{aligned}$$

where B_i is a polynomial in $(1/g)'$, $(1/g)''$, $\cdots$, $(1/g)^{(k)}$.

Since $\lim_{j\to\infty} g_j(\zeta_j)=\infty$, by (5) we get

$$\lim_{j\to\infty}\left[(k!)^q\left(\frac{g_j'(\zeta_j)}{g_j^2(\zeta_j)}\right)^{kq} g_j^{(k+1)q-1}(\zeta_j)+\sum_{i=1}^{(k+1)q-1} B_ig_j^{i-1}(\zeta_j)\right]=0.$$

Similarly, we have

$$\lim_{j\to\infty}\left[(k!)^q\left(\frac{g_j'(\zeta_j)}{g_j^2(\zeta_j)}\right)^{kq} g_j^{(k+1)q-2}(\zeta_j)+\sum_{i=1}^{(k+1)q-1} B_ig_j^{i-2}(\zeta_j)\right]=0.$$

Proceeding inductively, we obtain

$$\lim_{j\to\infty}\left[-\frac{g_j'(\zeta_j)}{g_j^2(\zeta_j)}\right]^k=0.$$

It follows that $(1/g(\zeta))'\big|_{\zeta=\zeta_0}=0$, so that ζ_0 is a multiple pole of $g(\zeta)$. Hence no pole of g is simple. This proves (ii).

It follows $Q(g^{(k)})\ne b$ and the definition of $Q(\omega)$ that there exist a non-zero constant c satisfying $g^{(k)}\ne c$. Hence by Lemma 2, we can deduce that g is a constant, which is a contradiction. Hence $\mathcal{F}$ is normal on D .

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