Average of two extreme points in JBW^*-triples

By Akhlaq Ahmad SIDDIQUI

Department of Mathematics, The Prince Sultan University,
P.O.Box 66833, Riyadh 11586, Saudi Arabia

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Abstract: H.Choda proved that every element in the closed unit ball of a von Neumann algebra is average of two extreme points of the ball. Here, we prove the strict generalisation of Choda's result to arbitrary JBW^*-triples.

Key words: von Neumann algebra; JB^*-algebra; JBW^*-algebra; JBW^*-triple; Peirce decomposition; extreme points.

A Jordan algebra $\mathcal{J}$ with product $\circ$ is called a Banach Jordan algebra if there is a norm $\|.\|$ on $\mathcal{J}$ making it a Banach space and $\|a \circ b\| \leq \|a\|\|b\|$. If, in addition, $\mathcal{J}$ has a unit e with $\|e\| = 1$ then $\mathcal{J}$ is called a unital Banach Jordan algebra. (details can be seen from [1,4,14,16], etc.)

In the sequel, $\{abc\}$ will denote the Jordan triple product of a, b, c defined in the Banach Jordan algebra $\mathcal{J}$ as follows: $\{abc\} = (a \circ b) \circ c - (a \circ c) \circ b + (b \circ c) \circ a$.

A complex Banach Jordan algebra $\mathcal{J}$ with involution $*$ (see [12], for instance) is called a JB^*-algebra if $\|\{xx^*x\}\| = \|x\|^3$ for all $x \in \mathcal{J}$. Let $\mathcal{J}$ be a JB^*-algebra with unit e. As usual, an element $u \in \mathcal{J}$ is called unitary if $u \circ u^* = e$ and $u^2 \circ u^* = u$; this simply means that u^* is the inverse of u.

The class of JB^*-algebras was introduced by Kaplansky in 1976 which includes all C^*-algebras as a proper subclass (see [18]). Around the same time, a related class called JB-algebras was being studied by Alfsen, Shultz and Størmer (see [1]): A real Banach Jordan algebra $\mathcal{J}$ is called a JB-algebra if $\|x\|^2 = \|x^2\| \leq \|x^2 + y^2\|$ for all $x, y \in \mathcal{J}$. These two classes of algebras are linked as follows (see [18,19]):

Theorem 1. (*a*). *If $\mathcal{A}$ is a JB^*-algebra then the set of self-adjoint elements (x with $x^* = x$) of $\mathcal{A}$ is a JB-algebra.*
(*b*). *If $\mathcal{B}$ is a JB-algebra then under a suitable norm the complexification $\mathcal{C}_{\mathcal{B}}$ of $\mathcal{B}$ is a JB^*-algebra.*

Analogous to the von Neumann algebras, JBW^*-algebras constitute an important subclass of JB^*-algebras: A JB^*-algebra is called a JBW^*-algebra if it is a Banach dual space. An example of a JBW^*-algebra can be constructed as follows:

Example 2. Let $\Re$ denote the field of real numbers. Let $\mathcal{H}$ be a real Hilbert space of dimension ≥ 2 with inner product $<,>$. The direct sum $\mathcal{X} := \mathcal{H} \oplus \Re$ is a JB-algebra with product

$$(h, \lambda) \circ (h', \lambda') = (\lambda h' + \lambda' h, < h, h' > + \lambda\lambda')$$

and norm

$$\|(h, \lambda)\| = \|h\| + |\lambda|$$

for all $h, h' \in \mathcal{H}$ and $\lambda, \lambda' \in \Re$. This JB-algebra is called a real spin factor and its complexification is a JBW^*-algebra (see [17]).

S. Sakai [13] studied von Neumann algebras from the viewpoint of C^*-algebra which is a Banach dual space and following his lines of argument one can easily see that every JBW^*-algebra $\mathcal{J}$ has a unique predual.

In fact, by using a different approach, Horn [6] extended some of this to the following class of subspaces of JB^*-algebras.

Let $\mathcal{A}$ be a JB^*-algebra and let $\mathcal{B}$ be a norm closed subspace of $\mathcal{A}$. If $\{xy^*z\} \in \mathcal{B}$ for all $x, y, z \in \mathcal{B}$ then $\mathcal{B}$ is called a JB^*-triple system (or JB^*-triple for short). If $\mathcal{B}$ is a JB^*-triple system which is a Banach dual space then $\mathcal{B}$ is called a JBW^*-triple system.

JB^*-triples are not algebras (as they generally are only closed under the triple product). Nevertheless they are associated with interesting geometric objects (see [10,16]) and it is possible for instance to calculate the extreme points of the unit ball of a JB^*-triple. If $\mathcal{X}$ is a normed space, we denote by $\mathcal{E}(\mathcal{X})_1$ the set of extreme points of the

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closed unit ball $(\mathcal{X})_1$ in $\mathcal{X}$.

Theorem 3 ([11]). *Let $\mathcal{J}$ be JB^*-triple and let $v \in \mathcal{J}$. $v \in \mathcal{E}(\mathcal{J})_1$ if and only if*

$$a - 2\{vv^*a\} + \{v\{va^*v\}^*v\} = 0$$

for all $a \in \mathcal{J}$.

When $\mathcal{J}$ is a JB^*-triple which is contained in the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on some Hilbert space $\mathcal{H}$, this condition can be written as $(e - vv^*)a(e - v^*v) = 0$ for all $a \in \mathcal{J}$, a condition originally obtained by Kadison [8] for C^*-algebras and later generalised by Harris [5]. If $v \in \mathcal{E}(\mathcal{J})_1$ then $v = \{vv^*v\}$, so v is a partial isometry (in the usual terms of operator theory), it is also frequently called tripotent in papers on triples; for details see [5–7,9,11].

If $\mathcal{J}$ is a JB^*-triple and v is a non-zero tripotent in $\mathcal{J}$ then $\mathcal{J}$ can be decomposed as folows:

$$\mathcal{J} = \mathcal{J}_0(v) \oplus \mathcal{J}_{\frac{1}{2}}(v) \oplus \mathcal{J}_1(v)$$

where $\mathcal{J}_k(v) = \{x \in \mathcal{J} : \{vv^*x\} = kx\}$ is called the Peirce k-space of v for $k = 0, \frac{1}{2}, 1$. The above decomposition of $\mathcal{J}$ is called the Peirce decomposition of $\mathcal{J}$ with respect to v; more details can be found in [16] where it is also shown that each Peirce space is a JB^*-subtriple of $\mathcal{J}$.

Let $\mathcal{J}$ be a JB^*-triple and let $v \in \mathcal{J}$ be a non-zero tripotent. v is called complete (or maximal) if $\mathcal{J}_0(v) = \{0\}$. In geometric terms, the complete tripotents of a JB^*-triple are precisely the extreme points of $(\mathcal{J})_1$ by [7; 1.15].

In Theorem 4 of paper [3], H. Choda proved that every element in the unit ball of a von Neumann algebra is the average of two extreme points of the unit ball. In the following discussion, we generalise this result of Choda to JBW^*-triples. This will be done in a completely different way to Choda's proof.

By [2, (2.2)], $\mathcal{J}_1(v)$ becomes a JB^*-algebra under product $\circ_1$ and the new involution $*_1$ defined by

$$x \circ_1 y = \{xv^*y\} \quad \text{and} \quad x^{*_1} = \{vx^*v\}.$$

If the tripotent u is unitary then $\mathcal{J}$ coincides with the Peirce 1-space; see details from [16].

We also need the following lemma for our main result:

Lemma 4. *If $\mathcal{J}$ is a JB^*-triple and $v \in \mathcal{E}(\mathcal{J})_1$, then every unitary u in the JB^*-algebra $\mathcal{J}_1(v)$ is in $\mathcal{E}(\mathcal{J})_1$.*

Proof. Since v an extreme point of $(\mathcal{J})_1$ is a complete tripotent, $\mathcal{J}_0(v) = \{0\}$. We denote the new triple product (induced by $\circ_1$) in $\mathcal{J}_1(v)$ by $\{,\,,\}_1$. Define the map F from the JB^*-triple $(\mathcal{J}_1(v), *, \{,,\})$ to the JB^*-triple $(\mathcal{J}_1(v), *_1, \{,,\}_1)$ by $F(y) = y$. Then F being a surjective isometry is a JB^*-triple isomorphism by [10, Proposition 5.5]. In particular, for any unitary u in the algebra $\mathcal{J}_1(v)$,

$$u = \{uu^{*_1}u\}_1 = \{uu^*u\}$$
$$\text{and} \quad \{uu^*v\} = \{uu^{*_1}v\}_1 = v.$$

This means u is a tripotent in $\mathcal{J}$ and $v \in \mathcal{J}_1(u)$. Hence by [6, Lemma 1.14(1)], $\mathcal{J}_0(u) \subseteq \mathcal{J}_0(v)$ so $\mathcal{J}_0(u) = \{0\}$. Thus $u \in \mathcal{E}(\mathcal{J})_1$. □

Theorem 5. *If $\mathcal{J}$ is a JBW^*-triple then every element of $(\mathcal{J})_1$ is the average of two extreme points of $(\mathcal{J})_1$.*

Proof. Let $x \in (\mathcal{J})_1$. By [6, Lemma 3.12], there exists a $v \in \mathcal{E}(\mathcal{J})_1$ such that $x \in \mathcal{J}_1(v)$ and $x = \{vx^*v\}$. Then $x^{*_1} = \{vx^*v\} = x$ so x is self-adjoint in the JB^*-algebra $\mathcal{J}_1(v)$. Hence there exist unitaries u_1, u_2 in the JB^*-algebra $\mathcal{J}_1(v)$ such that $x = \frac{1}{2}(u_1 + u_2)$ by [15, Theorem 2.11]. Then by the previous lemma $u_1, u_2 \in \mathcal{E}(\mathcal{J})_1$. Thus x is the average of two extreme points of $(\mathcal{J})_1$. □

Remark 6. Of course, in JB^*-triples which are not unital JB^*-algebras it is not possible to talk about unitary elements, so the above Theorem 5 is best possible for JB^*-triples. Apart from certain C^*-algebras (for example, von Neumann algebras) we do not have any example of infinite-dimensional JB^*-algebra in which every element of the unit ball is the average of 2 unitaries.

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