Hartogs-Osgood theorem for separately harmonic functions

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Abstract: Let h be a separately harmonic function on an open neighborhood of a $(m-1)$-dimensional compact submanifold Σ in $\mathbf{R}^m$ with $m \geq 2$. We show that h can be extended to a separately harmonic function on the bounded component of $\mathbf{R}^m - \Sigma$.

Key words: Separately harmonic; potential theory.

1. Introduction and main theorem.

A famous and fundamental theorem of Hartogs states that a separately holomorphic function, i.e. a holomorphic function with respect to each variable, is holomorphic. In particular, a separately holomorphic function is analytic.

Even in the case of real variables, there are various analogues of Hartogs theorem. For instance, Lelong showed in [6] a kind of Hartogs theorem for harminic functions.

Fact 1 (Lelong). Let $u(x,y)$ be defined on $B_1 \times B_2$, where B_1 is the unit ball in $\mathbf{R}^m$ and B_2 is the unit ball in $\mathbf{R}^n$. If $u(x,y)$ is separately harmonic, that is, $u(x,\cdot)$ is harmonic on B_2 for each x and $u(\cdot,y)$ is harmonic on B_1 for each y, then u is harmonic on $B_1 \times B_2$.

See also Avanissian [2], Siciak [8], Zaharjuta [12], Stein [9], Hervé [4] for related results.

Another famous and fundamental theorem, which shows a serious difference between $\mathbf{C}$ and $\mathbf{C}^n$ $(n \geq 2)$, is Hartogs-Osgood theorem.

Fact 2 (Osgood). Let $D \subset \mathbf{C}^n$ $(n \geq 2)$ be a domain, and K a compact subset of D such that $D - K$ is connected. Then every holomorphic function on $D - K$ can be extended to a holomorphic function on D.

We prove the following analogue of Hartogs-Osgood theorem for separately harmonic functions. First, we recall the definition of separately harmonic functions.

Definition. We say that a function $u : D \to \mathbf{R}$, where D is a domain on $\mathbf{R}^m = \mathbf{R}^{m_1} \times \cdots \times \mathbf{R}^{m_k}$, where $m_1, \ldots, m_k$ are fixed natural numbers, $m_1 + \cdots + m_k = m$, and $m \geq k \geq 2$, is *separately harmonic* on D if u is harmonic on each $\mathbf{R}^\nu$ $(\nu = m_1, \ldots, m_k)$ separately, i.e. the following identities hold on D:

$$\sum_{\nu=1}^{m_1} \frac{\partial^2 u}{\partial x_\nu^2} = 0, \quad \sum_{\nu=m_1+1}^{m_1+m_2} \frac{\partial^2 u}{\partial x_\nu^2} = 0, \ldots, \quad \sum_{\nu=m_1+\cdots+m_{k-1}+1}^{m} \frac{\partial^2 u}{\partial x_\nu^2} = 0.$$

In Section 2, we show:

Theorem 1. *Let D be a bounded domain in $\mathbf{R}^m$ with $m \geq 2$ whose boundary consists of a $(m-1)$-dimensional submanifold Σ. Let h be a separately harmonic function on an open neighborhood V of Σ in $\mathbf{R}^m$. Then h can be extended to a separately harmonic function on D.*

Compare Theorem 1 with the following fact in [3].

Fact 3 (Hecart). Let $D \subset \mathbf{R}^m$ and $G \subset \mathbf{R}^n$ be domains. Let $E \subset D$ and $F \subset G$ be compact sets which satisfy the Leja condition with respect to harmonic polynomials. Then there exists an open set $\Omega \subset \mathbf{R}^{m+n}$ such that each separately harmonic function $u : (D \times F) \cup (E \times G) \to \mathbf{R}$ extends to a harmonic function on Ω.

Remark. Fact 1 is false if "harmonic" is replaced by "subharmonic". Wiegerinck [10] gave an example $u(x,y)$ which is not subharmonic but separately subharmonic. On the other hand, some additional conditions for separately subharmonic functions to be subharmonic were given by many authors, for example, Riihentaus [7], Armitage and Gardiner [1].

We may ask whether u is subharmonic on $B_1 \times B_2$ if $u(x,\cdot)$ is harmonic on B_2 for each x and $u(\cdot,y)$ is subharmonic on B_1 for each y (where B_1 and B_2 are as in Fact 1). Kołodziej and Thorbiörnson [5]

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showed that the answer is yes if $u(\cdot, y)$ is of class C^2 for each y.

2. Proof of Theorem 1. Let $D, \partial D, V$ and h be as in Theorem 1. It is enough to show that, for a given integer k ($1 \leq k < m$), a separately harmonic function h on $V \subset \mathbf{R}^m = \mathbf{R}^k \times \mathbf{R}^{m-k}$ with $m \geq 2$ can be extended to a separately harmonic function on D, because separately harmonic functions are always harmonic. We need to consider only the case that $m \geq 3$: when $m = 2$, from the assumption,

$$h(x_1, x_2) = ax_1x_2 + bx_1 + cx_2 + d$$

on V with suitable constants a, b, c, and d. On the other hand, the right-hand side of the above equation is a separately harmonic function on $\mathbf{R}^2$, and hence h is extended to a separately harmonic function on $\mathbf{R}^2$.

Assume that $m \geq 3$. We take an open tubular neighborhood $W \Subset V$ of Σ, i.e. an open neighborhood of Σ diffeomorphic to $\Sigma \times (-1, 1)$ whose boundary consists of two smooth $(m-1)$-dimensional submanifold Σ_i ($i = 1, 2$) homotopic to $\pm\partial D$. We put

$$D_1 = D \cup W \text{ and } D_2 = \mathbf{R}^m - (D - W).$$

Then

$$D_1 \cup D_2 = \mathbf{R}^m,\ D_1 \cap D_2 = W \text{ and } \partial D_i = \Sigma_i\ (i = 1, 2).$$

Lemma 2. *There exist harmonic functions h_i on D_i $(i = 1, 2)$ such that*

$$h_2 - h_1 = h \text{ on } W. \tag{2.1}$$

Proof. Take open tubular neighborhoods T_i of Σ_i ($i = 1, 2$) in V such that $\overline{T_1} \cap \overline{T_2} = \emptyset$. Let $\chi_i \in C^\infty(\mathbf{R}^m)$ ($i = 1, 2$) such that $0 \leq \chi_i \leq 1$,

$$\chi_1(x) = \begin{cases} 1 & \text{on } T_1 \cup (\mathbf{R}^m - D_1) \\ 0 & \text{on } T_2 \cup (\mathbf{R}^m - D_2), \end{cases}$$

$$\chi_2(x) = \begin{cases} 0 & \text{on } T_1 \cup (\mathbf{R}^m - D_1) \\ 1 & \text{on } T_2 \cup (\mathbf{R}^m - D_2), \end{cases}$$

and

$$\chi_1 + \chi_2 = 1 \text{ on } \mathbf{R}^m.$$

If we extend $\chi_i h$ (by setting) to be 0 on $D_i - W$, then $\chi_i h \in C^\infty(D_i)$ ($i = 1, 2$). Further, since $\chi_i h = h$ on $T_i (\subset V)$, if we extend $\Delta(\chi_i h)$ (by setting) to be 0 on $\mathbf{R}^m - D_i$, then $\Delta(\chi_i h) \in C_0^\infty(\mathbf{R}^m)$ ($i = 1, 2$), where $\operatorname{Supp} \Delta(\chi_i h) \subset W - (T_1 \cup T_2) \Subset W$. Moreover, we have

$$\chi_1 h + \chi_2 h = h \text{ on } V,\ \Delta(\chi_1 h) + \Delta(\chi_2 h) = 0 \text{ on } \mathbf{R}^m. \tag{2.2}$$

Define

$$N_i(x) := c_m \int_{\mathbf{R}^m} \frac{\Delta(\chi_i h)(y)}{\|y - x\|^{m-2}} dV_y,\ x \in \mathbf{R}^m$$

for each $i = 1, 2$, where $c_m = \frac{1}{(m-2)\omega_m}$, ω_m is the euclidean surface area of the unit sphere in $\mathbf{R}^m$, and dV_y is the euclidean volume element of $\mathbf{R}^m$ at y. Set

$$\begin{aligned} h_1 &:= -(\chi_1 h + N_1) && \text{on } D_1, \\ h_2 &:= \chi_2 h + N_2 && \text{on } D_2. \end{aligned}$$

Then h_i ($i = 1, 2$) are harmonic functions on D_i satisfying (2.1). In fact, since N_i satisfy Poisson's equation, we have $\Delta N_i = -\Delta(\chi_i h)$ on $\mathbf{R}^m$, and hence h_i are harmonic on D_i.

By (2.2), we have $N_1 + N_2 = 0$ on $\mathbf{R}^m$, which implies the assertion. □

Remark. The argument as in the proof of the above lemma is used in various situations. See, for instance, [11].

Lemma 3. *The functions h_i defined above are separately harmonic functions on D_i $(i = 1, 2)$.*

Proof. We prove the assertion for $i = 1$, since the proof for $i = 2$ is exactly same.

Set

$$\tilde{\Delta}_1 := \sum_{\nu=1}^{k} \frac{\partial^2}{\partial x_\nu^2},\ \tilde{\Delta}_2 := \sum_{\nu=k+1}^{m} \frac{\partial^2}{\partial x_\nu^2}.$$

Fix a non-empty open set $U \Subset T_1 \cap D_1$ $(\subset W)$. Then we first show that

$$\tilde{\Delta}_1 h_1 = \tilde{\Delta}_2 h_1 = 0 \text{ on } U. \tag{2.3}$$

Take another open tubular neighborhood W_0 of Σ in W such that $\overline{U} \cap \overline{W_0} = \emptyset$. The boundary of W_0 consists of two smooth $(m-1)$-dimensional submanifold $\Sigma_{0,i}$ ($i = 1, 2$) with $\Sigma_{0,i} \subset T_i$ ($i = 1, 2$). Since $\Delta(\chi_1 h) \in C_0^\infty(\mathbf{R}^m)$, we have

$$\begin{aligned} -\tilde{\Delta}_j h_1(x_0) = {} & \tilde{\Delta}_j(\chi_1 h)(x_0) \\ & + c_m \int_{\mathbf{R}^m} \frac{\tilde{\Delta}_j\{\Delta(\chi_1 h)(y)\}}{\|y - x_0\|^{m-2}} dV_y \end{aligned}$$

for every $x_0 \in U$ and every $j = 1, 2$, where $\tilde{\Delta}_j$ in the integral is taken with respect to $(y_1, \ldots, y_k)$ when $j = 1$ and $(y_{k+1}, \ldots, y_m)$ when $j = 2$. Since $\chi_1 = 1$ on T_1 and $\operatorname{Supp} \Delta(\chi_1 h) \subset W - (T_1 \cup T_2) \Subset W_0$, it follows that

$$-\tilde{\Delta}_j h_1(x_0) = c_m \int_{W_0} \frac{\Delta\{\tilde{\Delta}_j(\chi_1 h)(y)\}}{\|y - x_0\|^{m-2}} dV_y$$

for $j = 1, 2$. Since $x_0 \notin W_0$, Green's formula gives that the right hand side of the above equality is

$$c_m \int_{\Sigma_{0,1}-\Sigma_{0,2}} \left(\frac{1}{\|y-x_0\|^{m-2}} \frac{\partial}{\partial n_y} (\tilde{\Delta}_j(\chi_1 h)) - (\tilde{\Delta}_j(\chi_1 h)) \frac{\partial}{\partial n_y} (\frac{1}{\|y-x_0\|^{m-2}}) \right) dS_y.$$

Here, $\chi_1 = 0$ on $T_2 \supset \Sigma_{0,2}$ and $\chi_1 = 1$ on $T_1 \supset \Sigma_{0,1}$, and hence this integral equals

$$c_m \int_{\Sigma_{0,1}} \left(\frac{1}{\|y-x_0\|^{m-2}} \frac{\partial}{\partial n_y} (\tilde{\Delta}_j h) - (\tilde{\Delta}_j h) \frac{\partial}{\partial n_y} (\frac{1}{\|y-x_0\|^{m-2}}) \right) dS_y.$$

Since h is separately harmonic on $V \supset \Sigma_{0,1}$, we have $\tilde{\Delta}_j h = 0$ $(j = 1, 2)$ on $\Sigma_{0,1}$. Therefore, the above integral is 0, which implies (2.3).

Next, we prove

$$\tilde{\Delta}_1 h_1 = \tilde{\Delta}_2 h_1 = 0 \text{ on } D_1. \tag{2.4}$$

Since h_1 is harmonic on D_1 by Lemma 2, h_1 is real-analytic on D_1. Hence $\tilde{\Delta}_1 h_1$ and $\tilde{\Delta}_2 h_1$ are also real-analytic on D_1. Then, by uniqueness of real analytic continuation, we have (2.4). □

Lemma 4. *The function h_2 is identically equal to* 0 *on D_2.*

Proof. Let D' be the projection of D_1 on the $\mathbf{R}^k$ of variables $x' = (x_1, \ldots, x_k)$, which is bounded in $\mathbf{R}^k$. Take a non-empty open set $U' \Subset \mathbf{R}^k - \overline{D'}$. For fixed $x'_0 \in U'$, let $L(x'_0)$ be the real $(m-k)$-dimensional plane $\{x'_0\} \times \mathbf{R}^{m-k}$ in $\mathbf{R}^m$. Since $L(x'_0) \subset D_2$, we have $\tilde{\Delta}_2 h_2(x'_0, x_{k+1}, \ldots, x_m) = 0$ on $L(x'_0)$. Since $N_2(x) = O(1/\|x\|)$ at $x = \infty$, by the maximum principle for harmonic functions, $h_2 = 0$ on $L(x'_0)$. Thus $h_2 = 0$ on $U' \times \mathbf{R}^{m-k}$. Again by uniqueness of real analytic continuation, we conclude the assertion. □

Thus by Lemma 4, we conclude that $h = -h_1$ on V, and hence $-h_1$ is the desired extension of h.

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