Geometric function theory and Smale's mean value conjecture

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Abstract: We improve an estimate of the constant in Smale's mean value conjecture, by using the Bieberbach theorem for coefficients of univalent functions and an estimate of the hyperbolic density of a certain simply connected domain.

Key words: Polynomial; critical point; univalent function; hyperbolic density.

1. Introduction and result. Let $P(z)$ be a complex polynomial of degree $d \geq 2$, and let z_1, z_2, …, z_{d-1} be the critical points of $P(z)$. Smale [11] stated that, if z is not a critical point of P, then the following inequality holds:

$$\min_i \left| \frac{P(z) - P(z_i)}{z - z_i} \right| \leq 4|P'(z)|. \tag{1}$$

Furthermore, he also formulated the following conjecture, which is known as Smale's mean value conjecture. See also [10] and [12].

Conjecture 1. Let $P(z)$ be a polynomial of degree $d \geq 2$ and let z_1, z_2, …, z_{d-1} be the critical points of $P(z)$. If z is not a critical point of P, then

$$\min_i \left| \frac{P(z) - P(z_i)}{(z - z_i)P'(z)} \right| \leq \frac{d-1}{d}. \tag{2}$$

A weaker version of Smale's conjecture is the inequality with constant 1 instead of $(d-1)/d$ in (2). Let $S(P, z)$ be the left-hand side of inequality (2), and denote by $K(d)$, $d \geq 2$, the smallest constant such that $S(P, z) \leq K(d)$ holds for all polynomials P of degree d and for all $z \neq z_i$. Inequality (1) shows that $K(d) \leq 4$ and in view of the example $P(z) = z^d - z$, one has $K(d) \geq (d-1)/d$. Smale's mean value conjecture thus says that $K(d) \leq (d-1)/d$. This conjecture has been proved only for degrees $d = 2, 3, 4$ (see [9]) and $d = 5$ (see [4]). For $d \geq 6$, it has been proved only under some additional conditions. See [7, 13, 14]. In a general case, Beardon, Minda and Ng [1] proved that

$$K(d) \leq 4^{\frac{d-2}{d-1}} =: K_1(d)$$

and Conte, Fujikawa and Lakic [2] verified that

$$K(d) \leq 4\,\frac{d-1}{d+1} =: K_2(d).$$

Furthermore, Schmeisser [8] showed that

$$K(d) \leq \frac{2^d - (d+1)}{d(d-1)} =: K_3(d).$$

In this paper, we improve these estimates.

Theorem 1. *Let P be a polynomial of degree $d \geq 2$ with critical points $z_1, z_2, \ldots, z_{d-1}$. If z is not a critical point of P, then*

$$\min_i \left| \frac{P(z) - P(z_i)}{(z - z_i)P'(z)} \right| \leq 4 \cdot \frac{1 + (d-2)4^{\frac{1}{1-d}}}{d+1} =: K_0(d).$$

Remark. For $d \geq 7$, our constant $K_0(d)$ is better than the other ones. More precisely,
(i) $K_0(d) < K_2(d) < K_1(d) < K_3(d)$ for $d \geq 8$;
(ii) $K_0(7) = 2.48425\ldots < K_3(7) < K_2(7) < K_1(7)$;
(iii) $K_3(d) < K_0(d) < K_2(d)$ for $d \leq 6$.
In particular, $K_3(6) = 1.9$. Note also that these results are superfluous when $d \leq 5$ since Smale's conjecture was already proved.

For all linear transformations α and β, we have $S(\beta \circ P \circ \alpha, \alpha^{-1}(z)) = S(P, z)$. Thus we have only to consider for $z = 0$ and for polynomials P satisfying $P(0) = 0$, $P'(0) = 1$ (see [1]). Namely, Smale's mean value conjecture is equivalent to the following

Conjecture 2. Let $P(z)$ be a polynomial of degree $d \geq 2$ with $P(0) = 0$ and $P'(0) = 1$, and let z_1, z_2, …, z_{d-1} be the critical points of $P(z)$. Then

$$\min_i \left| \frac{P(z_i)}{z_i} \right| \leq \frac{d-1}{d}.$$

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Conjecture 2 is called the normalized conjecture, and this has been proved for polynomials satisfying certain conditions. For example, either if all the critical points of P are real or if all the zeros of P but the origin have the same modulus, then the normalized conjecture is true. Furthermore, Ng [6] showed that $S(P,0) \leq 2$ for odd polynomials P. For a general case, we have the following, which is equivalent to Theorem 1.

Theorem 2. *Let $P(z)$ be a polynomial of degree $d \geq 2$ with $P(0)=0$ and $P'(0)=1$, and z_1, z_2, ..., z_{d-1} the critical points of $P(z)$. Then*

$$\min_i \left|\frac{P(z_i)}{z_i}\right| \leq K_0(d).$$

2. Proof of Theorem. We have only to prove Theorem 2. We denote by $\rho_\Omega(z)|dz|$ the hyperbolic metric of a hyperbolic domain Ω with curvature -4. The quantity $\rho_\Omega(z)$ is called the hyperbolic density of Ω at $z \in \Omega$. For instance, the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ has the hyperbolic density

$$\rho_{\mathbf{D}}(z) = \frac{1}{1-|z|^2}.$$

Lemma 1 ([1]). *For every domain Ω of the form $\mathbf{C} - (R_1 \cup \cdots \cup R_n)$ where R_i are rays of the form $\{re^{\mathrm{i}\theta_i} | \ r \geq 1\}$, the hyperbolic density $\rho_\Omega(z)$ of Ω satisfies the inequality*

$$\rho_\Omega(0) \leq 4^{-\frac{1}{n}}.$$

We will prove our theorem by using this lemma and the Bieberbach theorem for univalent functions on $\mathbf{D}$ (see [5]). The proof is based on the argument in [2].

Proof of Theorem 2. We may assume that $\min_i |z_i| = |z_1| = z_1 > 0$ and $\min_i |P(z_i)| = 1$ by compositions of linear transformations, see [2]. Then

$$\min_i \left|\frac{P(z_i)}{z_i}\right| \leq \left|\frac{P(z_j)}{z_j}\right| = \frac{1}{|z_j|} \leq \frac{1}{z_1},$$

where j is an integer satisfying

$$|P(z_j)| = \min_i |P(z_i)| = 1.$$

Thus we will prove that

$$\frac{1}{z_1} \leq K_0(d).$$

Since $z_1, \dots, z_{d-1}$ are the critical points of P and $P'(0)=1$, we have

$$P'(z) = \left(1 - \frac{z}{z_1}\right)\left(1 - \frac{z}{z_2}\right)\cdots\left(1 - \frac{z}{z_{d-1}}\right).$$

Then, since $P(0)=0$, this gives

$$P(z) = z - \left(\frac{1}{2}\sum_{i=1}^{d-1}\frac{1}{z_i}\right)z^2 + \left(\frac{1}{3}\sum_{i\neq j}^{d-1}\frac{1}{z_i z_j}\right)z^3 - \cdots + \frac{(-1)^{d-1}}{d\cdot z_1 z_2 \cdots z_{d-1}}\,z^d.$$

Let R_i be the ray of the form $\{re^{\mathrm{i}\theta_i} \mid r \geq 1\}$ that passes through $P(z_i)$. By Lemma 1, the hyperbolic density $\rho_\Omega(z)$ of $\Omega = \mathbf{C} - (R_1 \cup \cdots \cup R_{d-1})$ satisfies

$$\rho_\Omega(0) \leq 4^{-\frac{1}{d-1}}.$$

Since Ω does not contain any critical value of P, one can take a (single-valued) branch f of the inverse function P^{-1} on Ω so that $f(0)=0$. In this way, we obtain a univalent function

$$f : \Omega \to \mathbf{C} - \{z_1, \dots, z_{d-1}\}$$

such that $f(0)=0$ and $P(f(w)) = w$ for all $w \in \Omega$. Then f has the form

$$f(w) = w + a_2 w^2 + a_3 w^3 + \cdots.$$

Since f omits the value z_1 in Ω, the function

$$f_1(w) = \frac{f(w)}{1 - f(w)/z_1} = w + \left(a_2 + \frac{1}{z_1}\right)w^2 + \cdots$$

is analytic in Ω. By applying the Bieberbach theorem [5, Theorem 2.2] to the univalent function f_1 on $\mathbf{D}$ $(\subset \Omega)$, we have

$$\left|a_2 + \frac{1}{z_1}\right| \leq 2. \tag{3}$$

Since $P(f(w)) = w$, we obtain

$$-P''(0) = f''(0) = 2a_2.$$

Thus

$$a_2 = -\frac{P''(0)}{2} = \frac{1}{2}\sum_{i=1}^{d-1}\frac{1}{z_i}.$$

Therefore inequality (3) yields that

$$\left|\frac{3}{z_1} + \sum_{i=2}^{d-1}\frac{1}{z_i}\right| \leq 4.$$

Since we assumed that z_1 is real, we have

$$(4) \qquad \frac{3}{z_1} + \sum_{i=2}^{d-1} \operatorname{Re} \frac{1}{z_i} \le 4.$$

Let $\phi : \mathbf{D} \to \Omega$ be a conformal homeomorphism satisfying $\phi(0) = 0$, which has the form

$$\phi(\zeta) = c_1\zeta + c_2\zeta^2 + \cdots .$$

Since the hyperbolic density ρ_Ω of Ω satisfies

$$\rho_\Omega(\phi(\zeta))|\phi'(\zeta)| = \rho_{\mathbf{D}}(\zeta),$$

we have $\rho_\Omega(0)|c_1| = \rho_{\mathbf{D}}(0) = 1$. Thus

$$|c_1| = \frac{1}{\rho_\Omega(0)} \ge 4^{\frac{1}{d-1}}.$$

Consider the function

$$\begin{aligned} g(\zeta) &= (f \circ \phi)(\zeta) \\ &= c_1\zeta + (c_2 + c_1^2 a_2)\zeta^2 + \cdots , \end{aligned}$$

which maps $\mathbf{D}$ conformally into $\mathbf{C} - \{z_1, \dots, z_{d-1}\}$. Furthermore, for $i = 1, \cdots, d-1$, set

$$\begin{aligned} g_i(\zeta) &= \frac{g(\zeta)}{1 - g(\zeta)/z_i} \\ &= c_1\zeta + \left(c_2 + c_1^2\left(a_2 + \frac{1}{z_i}\right)\right)\zeta^2 + \cdots , \end{aligned}$$

and $h_i(\zeta) := g_i(\zeta)/c_1$. Then $h_i : \mathbf{D} \to \mathbf{C}$ is a univalent function satisfying $h_i(0) = 0$ and $h_i'(0) = 1$. By applying the Bieberbach theorem to $h_i(\zeta)$, we have

$$\left|\frac{c_2}{c_1} + c_1\left(a_2 + \frac{1}{z_i}\right)\right| \le 2,$$

namely,

$$\left|\frac{c_2}{c_1^2} + a_2 + \frac{1}{z_i}\right| \le \frac{2}{|c_1|}.$$

In particular,

$$\left|\frac{c_2}{c_1^2} + a_2 + \frac{1}{z_1}\right| \le \frac{2}{|c_1|}.$$

By the triangle inequality, we see that

$$\left|\frac{1}{z_i} - \frac{1}{z_1}\right| \le \frac{4}{|c_1|} \le 4 \cdot 4^{-\frac{1}{d-1}} = 4^{\frac{d-2}{d-1}}.$$

Since we assumed that z_1 is real, we have

$$(5) \qquad \frac{1}{z_1} - 4^{\frac{d-2}{d-1}} \le \operatorname{Re} \frac{1}{z_i}.$$

Therefore, inequalities (4) and (5) yield that

$$\frac{3}{z_1} + (d-2)\left(\frac{1}{z_1} - 4^{\frac{d-2}{d-1}}\right) \le 4.$$

This implies that

$$\frac{1}{z_1} \le 4 \cdot \frac{1 + (d-2)4^{\frac{1}{1-d}}}{d+1}$$

and we have proved our theorem. □

3. Concluding remark. The present framework can be used to show the existence of an extremal polynomial for the constant $K(d)$. Note that the existence of such a polynomial is not trivial. We end the article by showing the following proposition. Note that Crane [3, §5] gives essentially the same conclusion and our proof is similar to his argument.

Proposition 1. *Let d be an integer with $d \ge 2$. There exists a complex polynomial $P(z)$ of degree at most d such that $S(P, 0) = K(d)$.*

Proof. Denote by $\mathscr{P}_0(d)$ the set of complex polynomials $P(z)$ of degree d satisfying $P(0) = 0$, $P'(0) = 1$ and $\min_i |P(z_i)| = 1$, where $z_1, z_2, \dots, z_{d-1}$ are the critical points of $P(z)$. Recall then that $S(P, 0) = \min_i |P(z_i)/z_i|$. Set

$$\mathscr{P}(d) = \mathscr{P}_0(2) \cup \cdots \cup \mathscr{P}_0(d)$$

for $d \ge 2$. Our goal is to find a $P \in \mathscr{P}(d)$ such that $S(P, 0) = K(d)$.

First note that $K(d-1) \le K(d)$ for $d \ge 3$. Indeed, for each $P \in \mathscr{P}_0(d-1)$ define $P_n \in \mathscr{P}_0(d)$ so that $P_n'(z) = P'(z)(1 - z/n)$ for $n = 1, 2, \dots$. Then $S(P_n, 0) \to S(P, 0)$ as $n \to \infty$. Therefore, $K(d-1) \le K(d)$.

For each $P \in \mathscr{P}_0(d)$, we take a univalent function f on $\Omega = \mathbf{C} - (R_1 \cup \cdots \cup R_{d-1})$ with $f(0) = 0$ and $P \circ f = \mathrm{id}$ as in the proof of Theorem 2.

As we have seen in the last section, we have

$$K(d) = \sup_{P \in \mathscr{P}_0(d)} S(P, 0).$$

Therefore, there is a sequence P_n in $\mathscr{P}_0(d)$ such that $S(P_n, 0) \to K(d)$ as $n \to \infty$. Let f_n be the univalent function on Ω_n constructed above for $f = f_n$. The restriction of f_n to $\mathbf{D}$ is a member of the well-known family S of normalized univalent functions on the unit disk (cf. [5]). Since S is normal, we may assume that f_n converges to a function $f_\infty \in S$ uniformly on every compact subset of $\mathbf{D}$.

By the Koebe one-quarter theorem, $f(\mathbf{D})$ contains the disk $\Delta = \{|z| < 1/4\}$ for each $f \in S$. Thus we can define the inverse function f^{-1} of f on Δ. It is easy to see that $f_n^{-1} = P_n$ converges to f_∞^{-1} uniformly on every compact subset of Δ. If we write

$$P_n(z) = a_{n,0} + a_{n,1}z + \cdots + a_{n,d}z^d$$

and

$$f_\infty^{-1}(z) = a_0 + a_1 z + \cdots$$

around $z = 0$, the Cauchy integral formula gives

$$\begin{aligned} a_k &= \frac{1}{2\pi \mathrm{i}} \int_{|z|=1/8} \frac{f_\infty^{-1}(z)\mathrm{d}z}{z^{k+1}} \\ &= \lim_{n\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{|z|=1/8} \frac{P_n(z)\mathrm{d}z}{z^{k+1}} \\ &= \begin{cases} \lim_{n\to\infty} a_{n,k} & (0 \le k \le d) \\ 0 & (d < k). \end{cases} \end{aligned}$$

Therefore, f_∞^{-1} is the restriction of a polynomial Q of degree $\le d$ to Δ and P_n converges to Q uniformly on every compact subset of $\mathbf{C}$.

The degree of the limit polynomial Q is at least 2. Indeed, we take a critical point ζ_n of $P_n \in \mathscr{P}(d)$ so that $|P_n(\zeta_n)| = 1$. Since $K(d) \ge 1 - 1/d \ge 1/2$, we may assume that $S(P_n, 0) \ge 1/3$ for sufficiently large n. Since $S(P_n, 0) \le |P_n(\zeta_n)/\zeta_n| = 1/|\zeta_n|$, we have $|\zeta_n| \le 3$. Then we can take a subsequence so that ζ_n converges to a point ζ, which satisfies $Q'(\zeta) = 0$. In particular, $\deg Q \ge 2$.

Next we will prove that $S(Q, 0) = K(d)$. Let $\eta \ne 0$ be a critical point of Q such that $S(Q, 0) = |Q(\eta)/\eta|$. By the Hurwitz theorem, we can take a critical point η_n of P_n so that $\eta_n \to \eta$, and hence,

$$|P_n(\eta_n)/\eta_n| \to |Q(\eta)/\eta| = S(Q, 0).$$

Since

$$S(P_n, 0) \le |P_n(\eta_n)/\eta_n|$$

and

$$S(P_n, 0) \to K(d),$$

we have $S(Q, 0) \ge K(d)$. On the other hand,

$$S(Q, 0) \le K(\deg Q) \le K(d),$$

and thus, $S(Q, 0) = K(d)$. □

In the above proof, it seems difficult to exclude the possibility that $Q \in \mathscr{P}(d - 1)$. However, if we knew that $K(d-1) < K(d)$, then we could conclude that $Q \in \mathscr{P}(d)$. Note that Crane [3] pointed out that the assertion $K(d-1) < K(d)$ would lead to several conclusions concerning extremal polynomials.

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References

[1] A. F. Beardon, D. Minda and T. W. Ng, Smale's mean value conjecture and the hyperbolic metric, Math. Ann. **322** (2002), 623–632.
[2] A. Conte, E. Fujikawa and N. Lakic, Smale's mean value conjecture and the coefficients of univalent functions, Proc. Amer. Math. Soc. (to appear).
[3] E. Crane, Extremal polynomials in Smale's mean value conjecture, Comput. Methods Fucnt. Theory **6** (2006), 145–163.
[4] E. Crane, A computational proof of the degree 5 case of Smale's mean value conjecture. (Preprint).
[5] P. Duren, *Univalent functions*, Springer-Verlag, New York, 1983.
[6] T. W. Ng, Smale's mean value conjecture for odd polynomials, J. Aust. Math. Soc. **75** (2003), 409–411.
[7] Q. I. Rahman and G. Schmeisser, *Analytic theory of polynomials*, Oxford Univ. Press **26**, Oxford, 2002.
[8] G. Schmeisser, The Conjecture of Sendov and Smale, in *Approximation Theory* (B. Bojanov, ed.), DARBA, Sofia, 2002, 353–369.
[9] T. Sheil-Small, *Complex polynomials*, Cambridge Univ. Press, **75**, Cambridge, 2002.
[10] M. Shub and S. Smale, Computational complexity: on the geometry of polynomials and the theory of cost, SIAM J. Comput. **15** (1986), 145–161.
[11] S. Smale, The fundamental theorem of algebra and complexity theory, Bull. Amer. Math. Soc. (N. S.) **4** (1981), 1–36.
[12] S. Smale, Mathematical problem for the next century, Mathematics: Frontiers and Perspectives, eds. V. Arnold, M. Atiyah, P. Lax and B. Mazur, Amer. Math. Soc., 2000.
[13] D. Tischler, Critical points and values of complex polynomials, J. of Complexity **5** (1989), 438–456.
[14] D. Tischler, Perturbations of critical fixed points of analytic maps, in *Complex analytic methods in dynamical systems* (*Rio de Janeiro*, 1992) Astérisque **222** (1994), 407–422.