Painlevé VI transcendents which are meromorphic at a fixed singularity

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Abstract: We will study special solutions of the sixth Painlevé equation which are meromorphic at a fixed singularity. We will calculate the linear monodromy for our solutions. We will show the relation between Umemura's classical solutions and our solutions.

Key words: The Painlevé equation; monodromy data.

1. Introduction. The Painlevé equation can be represented by an isomonodromic deformation of a linear equation. We call the monodromy data of the linear equation *a linear monodromy* of the Painlevé function. In general the linear monodromy cannot be calculated explicitly, and we will study Painlevé functions whose linear monodromy can be explicitly determined. In this paper we call such Painlevé functions *monodromy solvable.*

Examples of monodromy solvable Painlevé functions are Umemura's classical solutions [10]. But there exist some monodromy solvable Painlevé functions which are not included in Umemura's classical solutions. It was R. Fuchs who first found a monodromy solvable solution that is not included in Umemura's classical solutions [2]. He calculated the linear monodromy of the so-called Picard's solution [9], which satisfies the sixth Painlevé equation

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2\\ &-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}\\ &+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\\ &\times\left[\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right]\end{aligned} \tag{1.1}$$

with special parameters: $\alpha=\beta=\gamma=0$ and $\delta=1/2$. R. Fuchs' result was discovered again recently [7]. Other monodromy solvable solutions are symmetric solutions of the first and second Painlevé equations which are shown by A. V. Kitaev [6]. The author has found the monodromy solvable solutios for the fourth and fifth Painlevé equations [3, 4].

In this paper we will classify the meromorphic solutions of the sixth Painlevé equation at each fixed singularity. There exist four meromorphic solutions at $t=0$ for generic values of parameters $\alpha,\beta,\gamma,\delta$. By some Bäcklund transformations we obtain four meromorphic solutions at each point $t=1$ and $t=\infty$. These twelve meromorphic solutions are invariant under the action of the Bäcklund transformation group.

The aim of this paper is to show that these meromorphic solutions are monodromy solvable. We will calculate the linear monodromy for one of these meromorphic solutions at $t=0$ by Jimbo's method [5]. We take two confluences of singularities of the linear equation. One is the confluence between $x=0$ and $x=t$ and the other is the confluence between $x=1$ and $x=\infty$. From these two confluences we obtain the linear monodromy for our solution explicitly.

One of our solutions includes the algebraic solution $y=\sqrt{t}$ for the parameters ($\alpha+\beta=0$, $\gamma+\delta=1/2$). Some of our solutions also include one of the Riccati solutions. In section five we describe the relations between our solutions and Umemura's classicial solutions.

2. Isomonodromic deformation of P_{VI}. The sixth Painlevé equation is represented by an isomonodromic deformation of a second order single equation [1, 8]:

$$\frac{\partial^2\psi}{\partial x^2}+p(x,t)\frac{\partial\psi}{\partial x}+q(x,t)\psi=0, \tag{2.1}$$

$$\frac{\partial\psi}{\partial t}=a(x,t)\frac{\partial\psi}{\partial x}+b(x,t)\psi, \tag{2.2}$$

where

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$$p(x,t)=\frac{1-\alpha_4}{x}+\frac{1-\alpha_3}{x-1}+\frac{1-\alpha_0}{x-t}-\frac{1}{x-y}, \tag{2.3}$$

$$q(x,t)=\frac{\alpha_2(\alpha_1+\alpha_2)}{x(x-1)}-\frac{t(t-1)H_{\mathrm{IV}}}{x(x-1)(x-t)}+\frac{y(y-1)z}{x(x-1)(x-y)}, \tag{2.4}$$

$$a(x,t)=\frac{y-t}{t(t-1)}\frac{x(x-1)}{x-y}, \tag{2.5}$$

$$b(x,t)=\frac{(1-\alpha_4-\alpha_3-\alpha_0)(y-t)}{2t(t-1)}-\frac{y(y-1)(y-t)z}{t(t-1)(x-y)}, \tag{2.6}$$

(2.7)

$$t(t-1)H_{\mathrm{VI}}=y(y-1)(y-t)z^2-[\alpha_4(y-1)(y-t)+\alpha_3 y(y-t)+(\alpha_0-1)y(y-1)]z+\alpha_2(\alpha_1+\alpha_2)(y-t).$$

The compatibility condition of (2.1) and (2.2) will give a Hamiltonian system

$$\frac{dy}{dt}=\frac{\partial H_{\mathrm{VI}}}{\partial z},\quad \frac{dz}{dt}=-\frac{\partial H_{\mathrm{VI}}}{\partial y}. \tag{2.8}$$

Eliminating z from the above system, we have the sixth Painlevé equation P_{VI} (1.1) for the parameters

(2.9)

$$\alpha=\frac{\alpha_1^2}{2},\quad \beta=-\frac{\alpha_4^2}{2},\quad \gamma=\frac{\alpha_3^2}{2},\quad \delta=\frac{1-\alpha_0^2}{2}.$$

3. Meromorphic solutions around a fixed singularity. In this section we will classify all of meromorphic solutions around a fixed singularity. We consider the solution of (2.8) around $t=0$:

$$y(t)=t^l\sum_{i=0}^{\infty}a_it^i,\quad z(t)=t^m\sum_{i=0}^{\infty}b_it^i, \tag{3.1}$$

$$(l,m\in\mathbf{Z},\ a_0\neq 0,\ b_0\neq 0).$$

Theorem 1. *For generic values of parameters, the sixth Painlevé equation has the following four meromorphic solutions around* $t=0$:

(3.2) (0-I) :
$$y(t)=\frac{\alpha_4}{\alpha_4-\alpha_0}t+O(t^2),\quad z(t)=\frac{\alpha_4-\alpha_0}{t}+O(t^0),$$

(3.3) (0-II) :
$$y(t)=\frac{\alpha_4}{\alpha_4+\alpha_0}t+O(t^2),\quad z(t)=\frac{\alpha_2(\alpha_1+\alpha_2)}{1-\alpha_4-\alpha_0}+O(t),$$

(3.4) (0-III) :
$$y(t)=\frac{\alpha_1+\alpha_3}{\alpha_1}+O(t),\quad z(t)=\frac{-\alpha_1\alpha_2}{\alpha_1+\alpha_3}+O(t),$$

(3.5) (0-IV) :
$$y(t)=\frac{\alpha_1-\alpha_3}{\alpha_1}+O(t),\quad z(t)=\frac{-\alpha_1(\alpha_1+\alpha_2)}{\alpha_1-\alpha_3}+O(t).$$

These solutions satisfy the system (2.8) and they are convergent since (2.8) is of Briot-Bouqet type at $t=0$. We gave a proof of the convergence for the fifth Painlevé transcendent in [4]. For generic values of parameters, there are no meromorphic solutions around $t=0$ except for these four solutions. The solution (0-I) exists for $\alpha_4-\alpha_0\notin\mathbf{Z}$.

Remark 2. In the case of $\alpha_1=0$ ($\alpha=0$), the sixth Painlevé equation has the following special solution around $t=0$:

(3.6)

$$y(t)=t^{-\alpha_3}(a_0+a_1t+a_2t^2+\cdots),\quad (a_i\in\mathbf{C}),$$

$$z(t)=t^{\alpha_3}(b_0+b_1t+b_2t^2+\cdots),\quad (b_i\in\mathbf{C}). \tag{3.7}$$

The Bäcklund transformations for the sixth Painlevé equation are shown in Table I.

Let σ_1 and σ_2 act on the solutions (0-I), (0-II), (0-III) and (0-IV). Then we obtain the meromorphic solutions of the system (2.8) which are meromorphic around $t=1$ and $t=\infty$.

Theorem 3. *The sixth Painlevé equation has the following meromorphic solutions*

(1) *around* $t=1$:

(3.8) (1-I):
$$y(t)=1+O\left((1-t)\right),\quad z(t)=\frac{\alpha_0-\alpha_3}{1-t}+O\big((1-t)^0\big),$$

(3.9) (1-II):
$$y(t)=1+O\left((1-t)\right),\quad z(t)=\frac{\alpha_2(\alpha_1+\alpha_2)}{\alpha_0+\alpha_3-1}+O((1-t)),$$

(3.10) (1-III):
$$y(t)=-\frac{\alpha_4}{\alpha_1}+O\left((1-t)\right),\quad z(t)=\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_4}+O((1-t)),$$

Table I. The Bäcklund transformations for the sixth Painlevé equation

x	α_0	α_1	α_2	α_3	α_4	y	z	t
$s_0(x)$	$-\alpha_0$	α_1	$\alpha_2+\alpha_0$	α_3	α_4	y	$z-\dfrac{\alpha_0}{y-t}$	t
$s_1(x)$	α_0	$-\alpha_1$	$\alpha_2+\alpha_1$	α_3	α_4	y	z	t
$s_2(x)$	$\alpha_0+\alpha_2$	$\alpha_1+\alpha_2$	$-\alpha_2$	$\alpha_3+\alpha_2$	$\alpha_4+\alpha_2$	$y+\dfrac{\alpha_2}{z}$	z	t
$s_3(x)$	α_0	α_1	$\alpha_2+\alpha_3$	$-\alpha_3$	α_4	y	$z-\dfrac{\alpha_3}{y-1}$	t
$s_4(x)$	α_0	α_1	$\alpha_2+\alpha_4$	α_3	$-\alpha_4$	y	$z-\dfrac{\alpha_4}{y}$	t
$\pi_1(x)$	α_3	α_4	α_2	α_0	α_1	$\dfrac{t}{y}$	$-\dfrac{y(yz+\alpha_2)}{t}$	t
$\pi_2(x)$	α_1	α_0	α_2	α_4	α_3	$\dfrac{(y-1)t}{y-t}$	$-\dfrac{z(y-t)^2+\alpha_2(y-t)}{t(t-1)}$	t
$\sigma_1(x)$	α_0	α_1	α_2	α_4	α_3	$1-y$	$-z$	$1-t$
$\sigma_2(x)$	α_0	α_4	α_2	α_3	α_1	$\dfrac{1}{y}$	$-y(yz+\alpha_2)$	$\dfrac{1}{t}$
$\sigma_3(x)$	α_4	α_1	α_2	α_3	α_0	$\dfrac{t-y}{t-1}$	$-(t-1)z$	$\dfrac{t}{t-1}$

$$\alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4=1$$

(3.11)

$$\text{(1-IV)}:\quad \begin{aligned} y(t)&=\frac{\alpha_4}{\alpha_1}+O\left((1-t)\right),\\ z(t)&=\frac{\alpha_1(\alpha_2+\alpha_4)}{\alpha_1-\alpha_4}+O((1-t)),\end{aligned}$$

(2) *around* $t=\infty$:

(3.12)

$$(\infty\text{-I}):\quad \begin{aligned} y(t)&=\frac{\alpha_1-\alpha_0}{\alpha_1}t+O\big((t^{-1})^0\big),\\ z(t)&=-\frac{\alpha_1(\alpha_1+\alpha_2)}{\alpha_1-\alpha_0}\cdot\frac{1}{t}+O\big(t^{-2}\big),\end{aligned}$$

(3.13)

$$(\infty\text{-II}):\quad \begin{aligned} y(t)&=\frac{\alpha_1+\alpha_0}{\alpha_1}t+O\big((t^{-1})^0\big),\\ z(t)&=-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_0}\cdot\frac{1}{t}+O\big(t^{-2}\big),\end{aligned}$$

(3.14)

$$(\infty\text{-III}):\quad \begin{aligned} y(t)&=\frac{\alpha_4}{\alpha_4+\alpha_3}+O\big(t^{-1}\big),\\ z(t)&=\frac{\alpha_2(\alpha_1+\alpha_2)}{1-\alpha_3-\alpha_4}\cdot\frac{1}{t}+O\big(t^{-2}\big),\end{aligned}$$

(3.15)

$$(\infty\text{-IV}):\quad \begin{aligned} y(t)&=\frac{\alpha_4}{\alpha_4-\alpha_3}+O\big(t^{-1}\big),\\ z(t)&=\alpha_4-\alpha_3+O\big(t^{-1}\big).\end{aligned}$$

Theorem 4. *These twelve meromorphic solutions are invariant under the action of the Bäcklund transformation group, which are shown in Fig.* 1.

4. The linear monodromy for the solution (0-I).

4.1. Normalization of (2.1). We transform the linear equation (2.1) so that the linear monodromy is a subgroup of $SL(2,\mathbf{C})$. By putting $\psi=x^{\alpha_4/2}(x-1)^{\alpha_3/2}(x-t)^{\alpha_0/2}(x-y)\tilde{\psi}$, (2.1) is transformed to

$$\frac{d^2\tilde{\psi}}{dx^2}+p_1(x,t)\frac{d\tilde{\psi}}{dx}+p_2(x,t)\tilde{\psi}=0, \tag{4.1}$$

where

$$p_1(x,t)=\frac{1}{x}+\frac{1}{x-t}+\frac{1}{x-1}+\frac{1}{x-y}, \tag{4.2}$$

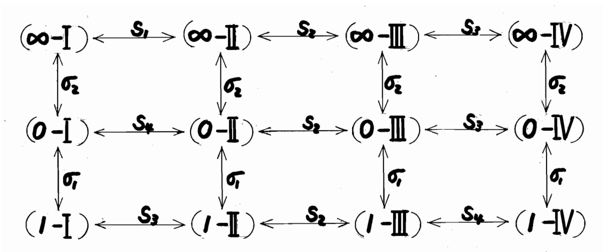

Fig. 1. The Bäcklund transformations of the twelve solutions.

(4.3)

$$p_2(x,t)=-\left(\frac{\alpha_4}{2}\right)^2\frac{1}{x^2}-\left(\frac{\alpha_3}{2}\right)^2\frac{1}{(x-1)^2}-\left(\frac{\alpha_0}{2}\right)^2\frac{1}{(x-t)^2}-\frac{1}{(x-y)^2}+\left(1-\frac{\alpha_4}{2}\right)\frac{1}{x(x-y)}+\left(\frac{\alpha_3}{2}+\frac{\alpha_4}{2}(1-\alpha_3)+\alpha_2(\alpha_1+\alpha_2)\right)\frac{1}{x(x-1)}+\left(\frac{\alpha_0}{2}+\frac{\alpha_4}{2}(1-\alpha_0)\right)\frac{1}{x(x-t)}+\left(\frac{\alpha_0}{2}+\frac{\alpha_3}{2}(1-\alpha_0)\right)\frac{1}{(x-1)(x-t)}+\left(1-\frac{\alpha_3}{2}\right)\frac{1}{(x-1)(x-y)}+\left(1-\frac{\alpha_0}{2}\right)\frac{1}{(x-t)(x-y)}-\frac{t(t-1)H_{\mathrm{IV}}}{x(x-1)(x-t)}+\frac{y(y-1)z}{x(x-1)(x-y)}.$$

The Riemann scheme of (4.1) is

(4.4)

$$P\begin{pmatrix} x=0 & x=t & x=1 & x=y & x=\infty & \\ \frac{-\alpha_4}{2} & \frac{\alpha_0}{2} & -\frac{\alpha_3}{2} & -1 & \frac{1}{2}(3+\alpha_1) & \\ \frac{\alpha_4}{2} & -\frac{\alpha_0}{2} & \frac{\alpha_3}{2} & 1 & \frac{1}{2}(3-\alpha_1) & ;x \end{pmatrix}.$$

We will calculate the linear monodromy $\{M_0, M_t, M_1, M_\infty\}$ of (4.1) for the solution (0-I) by the method in [5].

Here M_j $(j=0,t,1,\infty)$ is the monodromy matrix along the path around $x=j$, and

(4.5) $$M_\infty M_1 M_t M_0 = I_2.$$

4.2. The limit of (4.1). 4.2.1) We will take the limit $t\to 0$ after substituting the solution (0-I) into (4.1). The limit

(4.6) $$\tilde{\psi}_0(x)=\lim_{t\to 0}\tilde{\psi}(x,t)$$

satisfies

(4.7)

$$\frac{d^2\tilde{\psi}_0}{dx^2}+\left(\frac{3}{x}+\frac{1}{x-1}\right)\frac{d\tilde{\psi}_0}{dx}+\left[\left(1-\left(\frac{\alpha_0-\alpha_4}{2}\right)^2\right)\frac{1}{x^2}-\left(\frac{\alpha_3}{2}\right)^2\frac{1}{(x-1)^2}+\left(\frac{1}{2}(\alpha_0+\alpha_3+\alpha_4)-\frac{\alpha_3}{2}(\alpha_0+\alpha_4)+1-\alpha_0\alpha_4+\alpha_2(\alpha_1+\alpha_2)\right)\frac{1}{x(x-1)}\right]\tilde{\psi}_0=0.$$

The Riemann scheme of (4.7) is

(4.8) $$P\begin{pmatrix} x=0\cdot t & x=1 & x=\infty & \\ \frac{\alpha_0-\alpha_4}{2}-1 & -\frac{\alpha_3}{2} & \frac{1}{2}(3+\alpha_1) & \\ -\frac{\alpha_0-\alpha_4}{2}-1 & \frac{\alpha_3}{2} & \frac{1}{2}(3-\alpha_1) & ;x \end{pmatrix}$$

(4.9) $$= x^{((\alpha_0-\alpha_4)/2)-1}(x-1)^{-\alpha_3/2}\times P\begin{pmatrix} x=0\cdot t & x=1 & x=\infty & \\ 0 & 0 & \alpha_0+\alpha_1+\alpha_2 & \\ \alpha_4-\alpha_0 & \alpha_3 & \alpha_0+\alpha_2 & ;x \end{pmatrix}.$$

Therefore a fundamental system of solutions of (4.7) is

(4.10)

$$\big(x^{((\alpha_0-\alpha_4)/2)-1}(x-1)^{-\alpha_3/2}\times {}_2F_1(\alpha_0+\alpha_1+\alpha_2,\alpha_0+\alpha_2,1+\alpha_0-\alpha_4;x),\; x^{((\alpha_4-\alpha_0)/2)-1}(x-1)^{-\alpha_3/2}\times {}_2F_1(\alpha_1+\alpha_2+\alpha_4,\alpha_2+\alpha_4,1+\alpha_4-\alpha_0;x)\big).$$

The linear monodromy of (4.7) is equivalent to $\{M_tM_0, M_1, M_\infty\}$.

The exponent matrices of (4.7) at $x=0,1$ and ∞ are given by

(4.11) $$T_0=\begin{pmatrix} \frac{\alpha_0-\alpha_4}{2}-1 & 0 \\ 0 & -\frac{\alpha_0-\alpha_4}{2}-1 \end{pmatrix},\quad T_1=\begin{pmatrix} -\frac{\alpha_3}{2} & 0 \\ 0 & \frac{\alpha_3}{2} \end{pmatrix},$$

(4.12) $$T_\infty=\begin{pmatrix} \frac{3+\alpha_1}{2} & 0 \\ 0 & \frac{3-\alpha_1}{2} \end{pmatrix}.$$

We may assume

(4.13)

$$M_tM_0=e^{2\pi i T_0}=\begin{pmatrix} e^{\pi i(\alpha_0-\alpha_4)} & 0 \\ 0 & e^{-\pi i(\alpha_0-\alpha_4)} \end{pmatrix},$$

(4.14)

$$M_1=\Gamma_{01}^{-1}e^{2\pi i T_1}\Gamma_{01},\quad M_\infty=\Gamma_{0\infty}^{-1}e^{2\pi i T_\infty}\Gamma_{0\infty},$$

where

(4.15)

$$\Gamma_{01}=\begin{pmatrix} \frac{\Gamma(1+\alpha_0-\alpha_4)\Gamma(\alpha_3)}{\Gamma(1-\alpha_1-\alpha_2-\alpha_4)\Gamma(1-\alpha_2-\alpha_4)} & \frac{\Gamma(1+\alpha_4-\alpha_0)\Gamma(\alpha_3)}{\Gamma(1-\alpha_0-\alpha_1-\alpha_2)\Gamma(1-\alpha_0-\alpha_2)} \\ \frac{\Gamma(1+\alpha_0-\alpha_4)\Gamma(-\alpha_3)}{\Gamma(\alpha_0+\alpha_1+\alpha_2)\Gamma(\alpha_0+\alpha_2)} & \frac{\Gamma(1+\alpha_4-\alpha_0)\Gamma(-\alpha_3)}{\Gamma(\alpha_1+\alpha_2+\alpha_4)\Gamma(\alpha_2+\alpha_4)} \end{pmatrix},$$

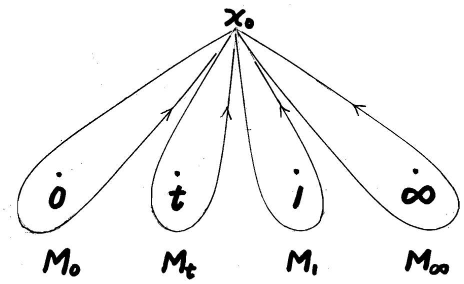

Fig. 2. The paths used to calculate the linear monodromies M_j.

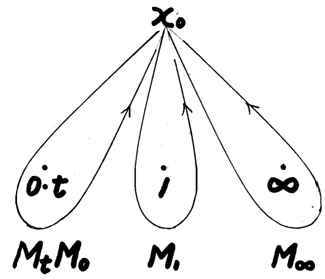

Fig. 3. The paths used to calculate the linear monodromies of (4.7).

(4.16)

$$\Gamma_{0\infty}=\begin{pmatrix} \dfrac{e^{(\alpha_0+\alpha_1+\alpha_2)\pi i}\Gamma(1+\alpha_0-\alpha_4)\Gamma(-\alpha_1)}{\Gamma(\alpha_0+\alpha_2)\Gamma(1-\alpha_1-\alpha_2-\alpha_4)} & \dfrac{e^{(\alpha_1+\alpha_2+\alpha_4)\pi i}\Gamma(1+\alpha_4-\alpha_0)\Gamma(-\alpha_1)}{\Gamma(\alpha_2+\alpha_4)\Gamma(1-\alpha_0-\alpha_1-\alpha_2)} \\ \dfrac{e^{(\alpha_0+\alpha_2)\pi i}\Gamma(1+\alpha_0-\alpha_4)\Gamma(\alpha_1)}{\Gamma(\alpha_0+\alpha_1+\alpha_2)\Gamma(1-\alpha_2-\alpha_4)} & \dfrac{e^{(\alpha_2+\alpha_4)\pi i}\Gamma(1+\alpha_4-\alpha_0)\Gamma(\alpha_1)}{\Gamma(\alpha_1+\alpha_2+\alpha_4)\Gamma(1-\alpha_0-\alpha_2)} \end{pmatrix}.$$

We should separate the monodromy data M_tM_0.

4.2.2) In the following, we consider the confluence of $x=1$ and $x=\infty$ in (4.1). We take the limit

$$\tilde{\psi}_1(\xi)=\lim_{t\to 0}\tilde{\psi}(t\xi,t). \tag{4.17}$$

Then $\tilde{\psi}_1(\xi)$ satisfies

(4.18)

$$\begin{aligned}&\frac{d^2\tilde{\psi}_1}{d\xi^2}+\left(\frac{1}{\xi}+\frac{1}{\xi-1}+\frac{1}{\xi-s}\right)\frac{d\tilde{\psi}_1}{d\xi}\\&+\Bigg[-\left(\frac{\alpha_4}{2}\right)^2\frac{1}{\xi^2}-\left(\frac{\alpha_0}{2}\right)^2\frac{1}{(\xi-1)^2}-\frac{1}{(\xi-s)^2}\\&\quad+\left(1-\frac{\alpha_4}{2}\right)\frac{1}{\xi(\xi-s)}+\left(\frac{\alpha_0}{2}+\frac{\alpha_4}{2}(1-\alpha_0)\right)\frac{1}{\xi(\xi-1)}\\&\quad+\left(1-\frac{\alpha_0}{2}\right)\frac{1}{(\xi-1)(\xi-s)}+\frac{\alpha_4(\alpha_0-1)}{\xi(\xi-1)}+\frac{\alpha_4}{\xi(\xi-s)}\Bigg]\tilde{\psi}_1\\&=0,\end{aligned}$$

where

$$s=\frac{\alpha_4}{\alpha_4-\alpha_0}. \tag{4.19}$$

This is a Heun's type equation with an apparent singularity at $\xi=s$. The singularities $\xi=0,1$ and ∞ correspond to $x=0,t$ and $1\cdot\infty$, respectively. The Riemann scheme of (4.18) is

$$P\begin{pmatrix} \xi=0 & \xi=1 & \xi=s & \xi=\infty & \\ -\dfrac{\alpha_4}{2} & \dfrac{\alpha_0}{2} & -1 & 1-\dfrac{\alpha_0-\alpha_4}{2} & \\ \dfrac{\alpha_4}{2} & -\dfrac{\alpha_0}{2} & 1 & 1+\dfrac{\alpha_0-\alpha_4}{2} & ;\xi \end{pmatrix}. \tag{4.20}$$

A fundamental system of solutions of (4.18) is

(4.21)

$$\left(\xi^{\alpha_4/2}(\xi-1)^{-\alpha_0/2}(\xi-s)^{-1},\ \xi^{-\alpha_4/2}(\xi-1)^{\alpha_0/2}(\xi-s)^{-1}\right).$$

The linear monodromy $\{L_0,L_1,L_\infty\}$ of (4.18) is equivalent to $\{M_0,M_t,M_\infty M_1\}$.

$$\begin{aligned}&M_0=P^{-1}L_0P,\quad M_t=P^{-1}L_1P,\\&M_\infty M_1=P^{-1}L_\infty P\end{aligned} \tag{4.22}$$

for a matrix $P\in SL(2,\mathbf{C})$.

The linear monodromy$\{L_0,L_1,L_\infty\}$ is given by

(4.23)

$$L_0=\begin{pmatrix} e^{-\pi i\alpha_4} & 0 \\ 0 & e^{\pi i\alpha_4}\end{pmatrix},\quad L_1=\begin{pmatrix} e^{\pi i\alpha_0} & 0 \\ 0 & e^{-\pi i\alpha_0}\end{pmatrix},$$

(4.24)

$$L_\infty=\begin{pmatrix} e^{-\pi i(\alpha_0-\alpha_4)} & 0 \\ 0 & e^{\pi i(\alpha_0-\alpha_4)}\end{pmatrix}.$$

Comparing (4.13) and (4.23), we have

(4.25)

$$\begin{aligned}&M_tM_0=P^{-1}L_1L_0P,\\&M_tM_0=L_1L_0=\begin{pmatrix} e^{\pi i(\alpha_0-\alpha_4)} & 0 \\ 0 & e^{-\pi i(\alpha_0-\alpha_4)}\end{pmatrix}.\end{aligned}$$

Thefore P is a diagonal matrix, since $\alpha_0-\alpha_4\notin\mathbf{Z}$ for the solution (0-I).

Theorem 5. *The linear monodromy of* (4.1) *for the solution* (0-I) *is as follows*:

(4.26)

$$M_0=\begin{pmatrix} e^{-\pi i\alpha_4} & 0 \\ 0 & e^{\pi i\alpha_4}\end{pmatrix},\quad M_t=\begin{pmatrix} e^{\pi i\alpha_0} & 0 \\ 0 & e^{-\pi i\alpha_0}\end{pmatrix},$$

$$\begin{aligned}&M_1=\Gamma_{01}^{-1}\begin{pmatrix} e^{-\pi i\alpha_3} & 0 \\ 0 & e^{\pi i\alpha_3}\end{pmatrix}\Gamma_{01},\\&M_\infty=\Gamma_{0\infty}^{-1}\begin{pmatrix} e^{\pi i\alpha_1} & 0 \\ 0 & e^{-\pi i\alpha_1}\end{pmatrix}\Gamma_{0\infty}.\end{aligned} \tag{4.27}$$

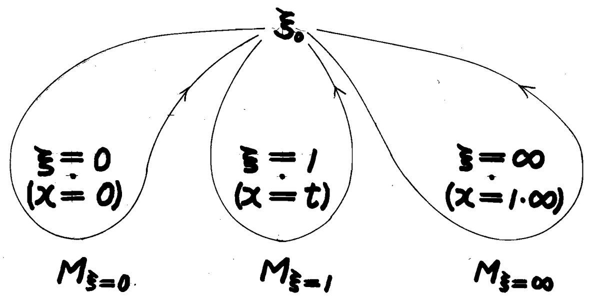

Fig. 4. The paths use to calculate the linear monodromies of (4.18).

Γ_{01} *and* $\Gamma_{0\infty}$ *are given in* (4.15) *and* (4.16). *We remark that* $\alpha_0 - \alpha_4 \notin \mathbf{Z}$ *if the solution* (0-I) *exists.* In a similar way, we can calculate the linear monodromy explicitly for all of the twelve solutions in Theorem 1 and Theorem 3.

Theorem 6. *The twelve solutions in Theorem* 1 *and Theorem* 3 *are all monodromy solvable.*

5. Comparison with classical solutions. Umemura studied special solutions of the Painlevé equations [10]. Umemura's classical solutions are either rational solutions or the Riccati solutions. We show that some of the twelve solutions of the sixth Painlevé equation include an algebraic solution and one of the Riccati solutions.

1) In the case of $\alpha_1 = \alpha_4$ and $\alpha_0 = \alpha_3$ ($\alpha + \beta = 0,\quad \gamma + \delta = 1/2$), the sixth Painlevé equation has an algebraic solution

(5.1)
$$y(t) = \sqrt{t} = 1 + \frac{1}{2}(t-1) + \frac{1}{2!}\cdot\frac{-1}{4}(t-1)^2 + \cdots.$$

The solution (5.1) is a special case of the solution (1-II) for $\alpha_1 = \alpha_4, \alpha_0 = \alpha_3$.

2) In the case of $\alpha_2 = 0$, the system (2.8) has the Riccati solution

(5.2)
$$z(t) \equiv 0,$$
$$y(t) = -\frac{t(t-1)}{\alpha_1} \times \frac{[(t-1)^{\alpha_4} {}_2F_1(\alpha_4, 1-\alpha_3, \alpha_0+\alpha_4; t)]'}{(t-1)^{\alpha_4} {}_2F_1(\alpha_4, 1-\alpha_3, \alpha_0+\alpha_4; t)} = -\frac{\alpha_4}{\alpha_1} + O(1-t).$$

This is obtained by putting $\alpha_2 = 0$ in the solution (1-III).

3) In the case of $\alpha_4 = 0$ ($\beta = 0$), the system (2.8) has the Riccati solution

(5.3)
$$y(t) \equiv 0,$$
$$z(t) = (t-1)\frac{[(t-1)^{\alpha_2} {}_2F_1(\alpha_2, \alpha_2+\alpha_3, 1-\alpha_0; t)]'}{(t-1)^{\alpha_2} {}_2F_1(\alpha_2, \alpha_2+\alpha_3, 1-\alpha_0; t)} = \alpha_2 + O(1-t).$$

This is obtained by putting $\alpha_4 = 0$ in the solution (1-III).

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