A discrete criterion in $PU(2,1)$ by use of elliptic elements

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Abstract: In this paper we show a 2-dimensional subgroup in $PU(2,1)$ which contains elliptics is discrete if and only if all its subgroups generated by two elliptics are discrete. This generalize the well-known discreteness criterion first established by T. Jørgensen.

Key words: Discrete groups; complex hyperbolic space; elliptic elements.

1. Introduction. The discreteness of Möbius groups is a fundamental problem, which have been discussed by many authors. In 1976, Jørgensen established his well-known result [12]:

Theorem A. *A non-elementary subgroup G of Möbius transformations acting on $\overline{\mathbf{R}}^2$ is discrete if and only if for each f and g in G the group $\langle f, g\rangle$ is discrete.*

This important result shows that the discreteness of a non-elementary Möbius group depends on the information of all its rank two subgroups. Furthermore, J. Gilman [7] and N. A. Isochenko [9] showed that the discreteness of all two-genrator subgroups, where each generator is loxodromic, is enough to secure the discreteness of the group.

For a space version of Theorem A, G. J. Martin [13] showed additional condition must be added, for example, the uniformly bounded torsion condition. In [6], Fang and Nai weakened it to Condition A, that is, there is no sequence $\{g_n\}$ of the involved group converging to the identity such that each g_n has more than two fixed points.

W. Abikoff and A. Hass in [1] constructed an example to show that for $n \geq 4$, there exist non-elementary subgroups of $\mathrm{Isom}(H^n)$ which are not discrete but their subgroups generated by finitely many elements are discrete. This means that in general Theorem A does not apply to space. They proved the following

Theorem B. *An n-dimensional subgroup Γ of $\mathrm{Isom}(H^n)$ is discrete if and only if every two-generator subgroup of Γ is discrete.*

Here by definition in [1] the n-dimensional conditional condition means that Γ does not have any Γ-invariant proper hyperbolic subspace. In addition, if n is even, they showed that Γ is discrete if and only if every two-generator subgroup of Γ is discrete.

In [5], Chen Min showed that for an n-dimensional subgroup G of $\mathrm{Isom}(H^n)$ and some fixed non-trivial Möbius transformation h, if for each $g \in G$ the group $\langle h, g\rangle$ is discrete, then G is discrete. The interesting thing is the test map h may be not in G.

In this paper, we discuss the generalization of Theorem A to the complex hyperbolic space. Denote by H^2_C the two dimensional complex hyperbolic space, and $PU(2,1)$ its holomorphic isometry group. Let G be a subgroup of $PU(2,1)$. Similar to [1] we give the following definition:

Definition 1. G is 2-dimensional if G doesn't leave invariant a point in $\overline{H^2_C}$ or a proper totally geodesic submanifold of H^2_C.

According to [4, Corollary 4.5.2], if G is a 2-dimensional subgroup of $PU(2,1)$ such that the identity is not an accumulation point of the elliptic elements in G, then G is discrete. A direct consequence is that a 2-dimensional subgroup containing no elliptics is discrete. So we are only interested in the case when the involved 2-dimensional group contains elliptic elements. The main purpose of this paper is to show the following result:

Theorem 1. *Let G be a 2-dimensional subgroup in $PU(2,1)$ and contains elliptic elements. Then G is discrete if and only if for each pair of elliptic elements f and g in G, the subgroup $\langle f, g\rangle$ is discrete.*

Jørgensen proved Theorem A by using the famous Jørgensen's inequality, which is a necessary condition for discreteness of two-generator groups. Recall that $PSL(2,\mathbf{R})$ can be identified with one dimensional complex hyperbolic group $PU(1,1)$. The

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generalization of Jørgensen's inequality to $PU(2,1)$ have been studied by A. Basmajian and R. Miner, Y. Jiang, J. Parker and S. Kamiya (See [2, 10, 11, 15, 16]). In this paper we use one of those generalization (cf. Coro. 11.1 in [2]) considering groups generated by two boundary elliptics, to prove our Theorem 1. The readers can refer to [8] for more about complex hyperbolic geometry.

2. Proof of the theorem. According to [14], a discrete subgroup G of $PU(2,1)$ is elementary if its limit set $L(G)$ contains at most two points and can be divided into the following three cases:

(a) elliptic type, i.e., $L(G) = \emptyset$. Then G is a finite group consisting of elliptics and all its elements share a common fixed point in H^2_C;

(b) parabolic type, i.e., $L(G) = \{a\}$. Then G consists of parabolic elements and probably elliptics with the fixed point a;

(c) loxodromic type, i.e., $L(G) = \{a, b\}$. Then G has a cyclic subgroup of finite index generated by a loxodromic element with fixed points a and b. If G contains an elliptic element, then it either fixes or exchanges a and b.

Lemma 1 ([2])**.** *Let f and g be boundary ellptic elements with fixed point chains C_1 and C_2 which are either linked or intersect at one point. Then there exists a positive real number ϵ so that if the group $\langle f, g\rangle$ is discrete, and f and g do not commute, then*

$$\max\{|\lambda(f) - 1|, |\lambda(g) - 1|\} > \epsilon.$$

Lemma 2. *If the two complex geodesics bounded by chains C_1 and C_2 intersect, then C_1 and C_2 are linked.*

Proof. Consider H^2_C as the ball model $\{(z_1, z_2) : |z_1|^2 + |z_2|^2 < 1\}$ with S^3 as its ideal boundary. Normalize so that the two complex geodesics intersect at the origin and C_1 consists of the points $\{(z_1, 0) : |z_1| = 1\}$. we may assume $C_2 = \{(z, az) \in S^3\}$, where a is a fixed complex number. Choose $q = (1, 0)$ as the pole. Then the ball model is mapped to the Siegal domain $\{(\omega_1, \omega_2) : 2Re(\omega_1) + |\omega_2|^2 < 0\}$ by Cayley transformation $(z_1, z_2) \mapsto (z_1/(1 + z_2), (1 - z_2)/(2(1 + z_2)))$. Denote by $\mathcal{H} = \mathbf{C} \times \mathbf{R}$ the Heisenberg space, whose one point compactification is the ideal boundary of the Siegal domain. We have the natural map from the ideal boundary of the Siegal domain to H which maps (z_1, z_2) to $(z_2/\sqrt{2}, \mathrm{Im}(z_1))$. Then the Heisenberg stereographic projection $\mathbf{P}\colon S^3 - \{q\} \mapsto \mathcal{H}$ maps (z_1, z_2) to $(z_2/(\sqrt{2}(z_1 - 1)), \mathrm{Im}(z_1 + 1)/2(z_1 - 1))$. Obviously, C_1 and C_2 correspond to the vertical axis and $(az/(\sqrt{2}(z - 1)), \mathrm{Im}(z + 1)/(2(z - 1)))$ in $\mathcal{H}$, respectively. Recall that the image of a finite chain under the vertical projection π to $\{(z, 0)\}$ is an Euclidean circle. Denote by x and r the center and radius of $\pi(C_2)$, respectively. Then we get the equality

$$|az - \sqrt{2}x(z - 1)|^2 = 2r^2\,|z - 1|^2\,.$$

Note that $|z|^2(|a|^2 + 1) = 1$, since $(z, az) \in S^3$. We can deduce that $2(|x|^2 - r^2) = \sqrt{2}a\overline{x}$. By combining the above equalities we obtain that $x = -\sqrt{2}/(2\overline{a})$ and $r^2 = 1 + 1/|a|^2$. Now it is easy to see that C_1 and C_2 are linked. □

Proof of Theorem 1. We only need to show the "if" part. Suppose that G is not discrete. Then there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of distinct elliptic elements such that $g_n \to I$ by [4, Corollary 4.5.2]. The proof can be divided into two cases.

Case 1. Each g_n is regular elliptic. Since $\langle g_m, g_k\rangle$ is discrete from the assumption, it is nilpotent for sufficently large m and k by Margulis Lemma and then elementary accoording to [3, Proposition 3.1.1]. Because regular elliptic elements have unique fixed point in H^2_C, $\langle g_m, g_k\rangle$ can not be a parabolic group. If $\langle g_m, g_k\rangle$ is of loxodromic type, g_m must swap two fixed points of some loxodrmic element in this group. Since $g_n \to I$, this is impossible if m and k are large enough. So we may assume all $\langle g_m, g_k\rangle$ is of elliptic type. Let $\mathrm{Fix}(\alpha)$ denote the set of points in $\overline{H^2_C}$ that are fixed by $\alpha \in PU(2,1)$. If α and β commute and $x \in \mathrm{Fix}(\beta)$, then $\alpha(x) = \alpha\beta(x) = \beta\alpha(x)$. It follows that $\alpha(\mathrm{Fix}(\beta)) = \mathrm{Fix}(\beta)$ and similarly $\beta(\mathrm{Fix}(\alpha)) = \mathrm{Fix}(\alpha)$. Hence g_m and g_k share the same fixed point if and only if they commute because each g_n is regular elliptic. Find an element γ of the 2-dimensional group G such that γ does not fix the common fixed point of g_n. Then both $\gamma g_n \gamma^{-1}$ and g_n are regular elliptic which converge to the identity as $n \to \infty$ but have different fixed point. By the same reason as above, $\langle \gamma g_n \gamma^{-1}, g_n\rangle$ is of elliptic type. This is a contradictition.

Case 2. Each g_n is boundary elliptic. Similarly, $\langle g_m, g_k\rangle$ is discrete and elementary for large m and k. If $\langle g_m, g_k\rangle$ is of elliptic type, g_m and g_k have a common fixed point in H^2_C. Then they commute by Lemma 1 and Lemma 2. If $\langle g_m, g_k\rangle$ is of parabolic or loxodromic type, we easily get g_m and g_k have a common fixed point in ∂H^2_C. Since the complex dilation factors $\lambda(g_n) \to 1$ as $n \to \infty$, g_m and g_k commute for sufficently large m and k by Lemma 1. So we may assume any two elements of $\{g_n\}$ commute. Note that two boundary elliptic elements commute if and only if they have either the same fixed point chains or the totally geodesic planes in H_C^2 bounded by these fixed point chains are orthogonal (See [2, p. 122]). Let $Fix_0(\alpha)$ denote the set of points in H_C^2 that are fixed by $\alpha \in PU(2,1)$. Then each $\mathrm{Fix}_0(g_n)$ is either the same as or orthogonal to $\mathrm{Fix}_0(g_1)$. Since each $\mathrm{Fix}_0(g_n)$ is a complex geodesic, the two complex geodesics, say $\mathrm{Fix}_0(g_{n_i})$ $(i = 1, 2)$, orthogonal to $\mathrm{Fix}_0(g_1)$ in H_C^2 are either the same or parallel. Note that $\{g_{n_i}\}$ also commute. Then $\mathrm{Fix}_0(g_{n_i})$ $(i = 1, 2)$ must coincide. Thus we may pick out a subsequence of g_n, which is still denoted by $\{g_n\}$, such that each element shares the same fixed point set, which we denoted by π. Claim that there exist two points $x, y \in L(G)$ which are not contained in π. First, There must be such a point, say x. Otherwise, $L(G) \subset \pi$. Since G keeps $L(G)$ invariant and each element in $PU(2,1)$ preserves complex geodesics, it follows that π is invariant under G. This is a contradiction to the 2-dimensional condition. Next, assume that there is only one such a point, that is, $L(G) = \{x\} \cup S$, where $S \subset \pi$. Since $g(L(G)) = L(G)$ and g preserves complex geodesics for each $g \in G$, we must have $g(S) = (S)$ and then $g(x) = x$. This is also contradict to the 2-dimensional condition. Let U and V be neighbourhoods of x and y which do not intersect with π, respectively. Thus there is a loxodromic $\gamma \in G$ with one fixed point in U and the other in V by [17, Theorem 2R]. For p sufficiently large, $\gamma^p(\pi) \subset U$ and then $\gamma^p(\pi) \cap \pi = \emptyset$. Hence $\gamma^p g_n \gamma^{-p}$ and g_n do not commute. By the same procedure, it follows that $\langle \gamma^p g_n \gamma^{-p}, g_n \rangle$ is discrete and elementary for all large n and then $\gamma^p g_n \gamma^{-p}$ and g_n commute. This is a contradiction. □

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