Invariant theoretical characterization of toric locally complete intersection singularities

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Abstract: In the previous paper [N3], we introduced a cofree embedding of a finite dimensional rational representation of a diagonalizable algebraic group over an algebraically closed field of characteristic zero. Using this, we show inductive characterization of toric locally complete intersection singularities in terms of group representation theory. Consequently we obtain a classification of affine monomial normal hypersurfaces.

Key words: Cofree representations; algebraic tori; character groups; toric singularities; complete intersections; monomial hypersufaces.

1. Introduction. Let G always denote a diagonalizable affine algebraic group over an algebraically closed field K of characteristic zero. Let $\mathfrak{X}(G)$ denote the rational character group of G over K which is regarded as an additive group. For an affine variety X over K, $\mathcal{O}(X)$ stands for the K-algebra of all polynomial functions on X. Moreover, for a regular action of G on an affine variety X (abbr. (X, G)), we denote by $X//G$ the algebraic quotient of X under the action of G with the quotient map $\pi_{X,G} : X \to X//G$. The action (X, G) is said to be *cofree* (resp. *stable*), if $\mathcal{O}(X)$ is free as an $\mathcal{O}(X)^G$-module (resp. X contains a non-empty open subset consisting of closed G-orbits). Let X_{st} denote the affine variety defined by $\mathcal{O}(X_{\mathrm{st}}) = \mathcal{O}(X)_{\mathrm{st}}$, where $\mathcal{O}(X)_{\mathrm{st}}$ denote the K-subalgebra of $\mathcal{O}(X)$ generated by $\mathcal{O}(X)_\chi$'s, $\chi \in \mathfrak{X}(G^0)$, satisfying $\mathcal{O}(X)_\chi \cdot \mathcal{O}(X)_{-\chi} \neq \{0\}$. In the case where X is normal, for a prime ideal $\mathfrak{P}$ such that $\mathrm{ht}(\mathfrak{P}) = \mathrm{ht}(\mathfrak{P} \cap \mathcal{O}(X)^G) = 1$ let $I_G(\mathfrak{P})$ denote the inertia group at $\mathfrak{P}$ and $\mathrm{e}(\mathfrak{P}, \mathfrak{P} \cap \mathcal{O}(X)^L)$ the ramification index of $\mathfrak{P}$ over $\mathfrak{P} \cap \mathcal{O}(X)^G$ (cf. [N2]).

If Λ is a subset of $\mathfrak{X}(G)$, let $\mathbf{Z}_0 \cdot \Lambda$ (resp. $\mathbf{Z} \cdot \Lambda$) denote the set of all linear combinations of any finite subset of Λ with coefficients in $\mathbf{Z}_0$ (resp. $\mathbf{Z}$) in $\mathfrak{X}(G)$, where $\mathbf{Z}_0$ denotes the additive semigroup of nonnegative integers and $\mathbf{Z}_0 \cdot \emptyset$ means $\{0\}$. Consider a finite dimensional rational G-module V. Let $V^\vee$ be the dual module of V on which G acts naturally and $\mathcal{W}(V, G)$ denote the set of all weights of G on V (i.e., $\{\chi \in \mathfrak{X}(G) \mid V_\chi \neq \{0\}\}$). We say that $\sigma \in G$ is a pseudo-reflection on V, if $\dim(\sigma - 1)(V^\vee) = \mathrm{ht}((\sigma - 1)(V^\vee) \cdot \mathcal{O}(V) \cap \mathcal{O}(V)^G) \leq 1$. Let $\mathcal{R}_V(G)$ denote the subgroup of G generated by all pseudo-reflections of G on V.

Definition 1.1. A pair (W, w) is defined to be a *paralleled linear hull* of (V, G), if W is a G-submodule of V_{st} and $0 \neq w \in V_{\mathrm{st}}$ such that $W \cap < G \cdot w >_K = \{0\}$ and the G_w-equivariant morphism

$$(\bullet + w) : W \ni x \mapsto x + w \in V_{\mathrm{st}}$$

induces the isomorphism $\pi_{V_{\mathrm{st}}//G_w, V//G} \circ (\bullet + w)//G_w : W//G_w \xrightarrow{\sim} V_{\mathrm{st}}//G$. Here $(\bullet + w)//G_w : W//G_w \longrightarrow V_{\mathrm{st}}//G_w$ is the quotient of $(\bullet + w)$ modulo G_w and $\pi_{V_{\mathrm{st}}//G_w, V//G} : V_{\mathrm{st}}//G_w \longrightarrow V_{\mathrm{st}}//G$ is defined by the inclusion $\mathcal{O}(V_{\mathrm{st}})^G \hookrightarrow \mathcal{O}(V_{\mathrm{st}})^{G_w}$. A paralleled linear hull (W_0, w_o) of (V, G) is said to be *minimal*, if W_0 is minimal with respect to inclusions in the set consisting of all subspaces W's of V_{st} such that (W, w)'s are paralleled linear hulls of (V, G) for some w's.

1.2. Toric singularities. Let $U_\sigma = \mathrm{Spec}(K[M \cap \sigma^\vee])$ be an affine toric variety associated to a rational strongly convex polyhedral cone σ. Then there exists a diagonalizable G and a finite dimensional rational representation $\rho : G \to GL(V)$ such that

$$K[M \cap \sigma^\vee] \cong K[V]^G$$

which preserves homogeneous parts. For any U_σ, let $\mathfrak{R}(U_\sigma)$ denote the set consisting of all pairs (V, G) as above. We put

$$\mathfrak{d}((V, G)) = (\dim V, \mathrm{rk}(\rho(G)), |\rho(G)/\rho(G^0)|) \in \mathbf{Z}^3$$

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and define the lexicographical order in $\mathbf{Z}^3$ on the subset $\mathfrak{d}(\mathfrak{R}(U_\sigma)) = \{\mathfrak{d}((V,G)) \mid (V,G) \in \mathfrak{R}(U_\sigma)\}$. By (2) of (1.1) of [N3], we see that, for $(V,G) \in \mathfrak{R}(U_\sigma)$, a minimal paralleled hull (W,w) of (V,G) is characterized by

$$\mathfrak{d}((W//\mathcal{R}_W(G_w), G_w/\mathcal{R}_W(G_w))) = \min\ (\mathfrak{d}(\mathfrak{R}(U_\sigma))).$$

Here $(W//\mathcal{R}_W(G_w), G_w/\mathcal{R}_W(G_w))$ is the natural representation, because $\mathcal{R}_W(G_w)$ acts as a finite group on W (cf. [N2]) and $W//\mathcal{R}_W(G_w)$ is an affine space (cf. [S]). So the ring theoretical properties on toric singularities should be described in terms of representation theory of (W,G_w)'s which are minimal in $\mathfrak{R}(U_\sigma)$ modulo pseudo-reflection parts.

Definition 1.3 (cf. [N3]). For a finite dimensional representation $\phi : H \to GL(W)$ of a diagonalizable group H, a faithful rational representation $\widetilde{\phi} : \widetilde{H} \to GL(W)$ of a diagonalizable group $\widetilde{H}$ is said to be a *cofree embedding of* $\phi : H \to GL(W)$ (or *of* (W,H)), if the following conditions (1), (2) are satisfied:

(1) $\phi(H)$ $(= H|_W) \subseteq \widetilde{\phi}(\widetilde{H})$ and $\phi(\mathcal{R}_W(H)) = \widetilde{\phi}(\mathcal{R}_W(\widetilde{H}))$.

(2) The representation $\widetilde{\phi}$ is stable and cofree.

A cofree embedding $\widetilde{\phi} : \widetilde{H} \to GL(W)$ of $\phi : H \to GL(W)$ is said to be *canonical*, if $\widetilde{\phi}(\widetilde{H})$ is minimal in all $\psi(L)$'s, where $\psi : L \to GL(W)$ are cofree embeddings ϕ.

In Sect. 2, we will show the ladder of subgroups between G_w and its cofree embedding $\widetilde{G_w}$ under the assumption that $V//G$ are complete intersections. We finally in Sect. 3 give a criterion for $V//G$ to be a hypersurface in terms of character groups of G, which can be regarded as a generalization of D. L. Wehlau's criterion (cf. [W]) for $V//G$ to be an affine space.

The symbol $\sharp(\ \circ\ )$ stands for the cardinality of the set $\circ$ and, for $n \in \mathbf{N}$, let $\mathbf{Z}_0^n$ denote the additive submonoid $\{(a_1,\ldots,a_n) \mid \forall a_i \in \mathbf{Z}_0\}$ of $\mathbf{Z}^n$. For a mapping $\varphi : A \to B$ and a subset $A' \subseteq A$, let $\varphi|_{A'}$ denote the restriction of φ to A' and, for a set Ω of mappings $\varphi : A \to B$, let $\Omega|_{A'}$ be the set of restrictions $\varphi|_{A'}$'s $(\varphi \in \Omega)$.

2. Ladder of subgoups of cofree embeddings and complete intersection singularities.

In this section, we study on the relation between $V//G$ and $\mathfrak{X}(G)$ under the following circumstances: Let V be a finite dimensional rational G-module. Fix a minimal paralleled linear hull (W,w) of (V,G) with $H = G_w$ and let $\{X_1,\ldots,X_m\}$ be a K-basis of the dual space $W^\vee$ of W on which H is diagonal. For a monomial M in $\mathcal{O}(W)$ of $\{X_k\}$, let $\mathrm{supp}(M)$ be the set

$$\{X_i \mid 1 \le i \le m, \mathcal{O}(W)\cdot X_i \ni M\}.$$

Let $D_{\{X_k\}}(W)$ denote the subgroup of $GL(W)$ consisting of all diagonal matrices on the K-basis $\{X_1, \ldots, X_m\}$ of W.

In the case where $V//G$ is a complete intersections, we show in [N3] that there exists a cofree embedding of (W,H) and will construct a ladder of subgroups of $GL(W)$ from H to its cofree embeddings as follows:

Theorem 2.1. *Suppose that $V//G$ is a complete intersection. For a minimal paralleled linear hull (W,w) of (V,G), let $\{X_1,\ldots,X_m\}$ be a K-basis of $W^\vee$ on which $H = G_w$ is diagonal. Then, for any canonical cofree embedding $(W,\widetilde{H})$ of (W,H) diagonal on this basis, there exists a descending chain of closed subgroups*

$$\widetilde{H} = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_d = H|_W$$

of $GL(W)$ such that each $\mathcal{O}(W)^{H_i}$, $0 \le i < d$, is generated by a part of the unique minimal system of generators of $\mathcal{O}(W)^{H_{i+1}}$ consisting of monomials of $\{X_1,\ldots,X_m\}$ as a K-algebra and is a complete intersection of $\mathrm{codim}(\mathcal{O}(W)^{H_i}) = i$.

2.2. Semigroup rings. For an affine additive monoid $\mathcal{S}$, we denote by $K[\mathcal{S}]$ the affine semigroup ring associated with $\mathcal{S}$ over K, i.e., $K[\mathcal{S}]$ is the K-vector space with the basis $\{\mathbf{e}(s) \mid s \in \mathcal{S}\}$ whose K-algebra structure is induced by $\mathbf{e}(s)\cdot\mathbf{e}(t) = \mathbf{e}(s+t)$ $s, t \in \mathcal{S}$ (e.g., [TE]). Let $\mathrm{FUND}(\mathcal{S})$ denote the minimal system of generators of $\mathcal{S}$ as a monoid and put $\mathcal{S}^\sharp = \mathcal{S}\backslash(\{0\} \cup \mathrm{FUND}(\mathcal{S}))$. For any $n \in \mathbf{Z}_0$ and nonzero $x \in \mathcal{S}$, let $\mathcal{S}\int_n x$ denote the submonoid

$$\mathcal{S} + \sum_{i=1}^n \mathbf{Z}_0 \cdot e_i + \mathbf{Z}_0 \cdot \left(x - \sum_{i=1}^n e_i\right)$$

of $\mathcal{S} \oplus \mathbf{Z}^n$, where e_i, $1 \le i \le n$, form the canonical basis of $\mathbf{Z}^n$. In the case where $n = 0$, the monoid $\mathcal{S}\int_0 x$ is regarded as $\mathcal{S}$. It should be noted that $\mathcal{S}$ is normal if and only if so is $\mathcal{S}\int_n x$. Clearly $\mathrm{FUND}(\mathcal{S}\int_n x) = \mathrm{FUND}(\mathcal{S}) \cup \{e_i \mid 1 \le i \le n\} \cup \{x - \sum_{i=1}^n e_i\}$.

Theorem 2.2.1 ([N1]). *For an affine normal monoid $\mathcal{S}$, the semigroup ring $K[\mathcal{S}]$ is a complete intersection of codimension d if and only if $\mathcal{S}$ is isomorphic to $(\cdots((\mathbf{Z}_0^{n_0}\int_{n_1} x_1)\int_{n_2} x_2)\int_{n_3}\cdots)\int_{n_d} x_d \oplus \mathbf{Z}^u$, where $n_i \in \mathbf{N}$, $u \in \mathbf{Z}_0$ and $x_i \in ((\cdots((\mathbf{Z}_0^{n_0}\int_{n_1} x_1)\int_{n_2} x_2)\int_{n_3}\cdots)\int_{n_{i-1}} x_{i-1})^\sharp$.*

Suppose that $V//G$ is a complete intersection of $\mathrm{codim}(\mathcal{O}(V)^G) = d$. Since $\mathcal{O}(W)^H$ is regarded as an affine semigroup ring $K[\mathcal{S}]$ without non-trivial units induced by the multiplicative monoid $\{\prod_{i=1}^m X_i^{c_i} \in \mathcal{O}(W)^H \mid \exists c_i \in \mathbf{Z}_0\}$ of monomials isomorphic to the additive monoid $\mathcal{S}$. Let $K[T]$ be the polynomial ring $K[T_{ij} \mid 0 \le i \le d,\ 0 \le j \le n_i\ (i \ge 1),\ 1 \le j \le n_0\ (i = 0)]$ with indeterminate $\{T_{ij}\}$ over K. Then let

$$\Phi : K[T] \to K[\mathcal{S}]$$

be a K-epimorphism defined by $\{\Phi(T_{i'j}) \mid 0 \le i' \le i,\ 0 \le j \le n_{i'}\ (i' \ge 1),\ 1 \le j \le n_0\ (i' = 0)\} = \mathrm{FUND}((\cdots((\mathbf{Z}_0^{n_0} \int_{n_1} x_1) \int_{n_2} x_2) \int_{n_3} \cdots) \int_{n_i} x_i)$ $(0 \le i \le d)$. By Proposition 4.1 of [N1], *the element* $\prod_{j=1}^{n_0} \Phi(T_{0j})$ *principally generates* $\omega_{K[\mathcal{S}]}$ *as an ideal.* The restriction of Φ induces the K-epimorphism $\Phi_i : K[T_{i'j} \mid 0 \le i' \le i,\ 0 \le j \le n_i'\ (i' \ge 1),\ 1 \le j \le n_0\ (i' = 0)] \to K[(\cdots((\mathbf{Z}_0^{n_0} \int_{n_1} x_1) \int_{n_2} x_2) \int_{n_3} \cdots) \int_{n_i} x_i]$ whose kernel is generated by

$$\left\{ \prod_{j=0}^{n_{i'}} T_{i'j} - \xi_{i'} \,\middle|\, 1 \le i' \le i \right\}$$

for $1 \le i \le d$, where $\xi_{i'}$ is a monomial of $\{T_{i''j} \mid 0 \le i'' < i',\ 0 \le j \le n_{i''}\ (i'' \ge 1),\ 1 \le j \le n_0\ (i'' = 0)\}$ such that $\Phi(\xi_{i'}) = \mathbf{e}(x_{i'})$. Denote by M_{ij} the image of T_{ij} under the composite $\Theta \circ \Phi$, where Θ stands for the canonical isomorphism $K[\mathcal{S}] \xrightarrow{\sim} \mathcal{O}(W)^H$. By the observation as above, we see that each

$$K[M_{i'j} \mid i' \le i,\ \forall j]$$

is normal. Let H_i $(0 \le i \le d)$ be the stabilizer of $D_{\{X_k\}}(W)$ at the set $\{M_{i'j} \mid 0 \le i' \le i, \forall j\}$. Let e_i be the order of $I_H(\mathcal{O}(W) \cdot X_i)|_W$, $1 \le i \le m$ ([N2]). Then, by the proof of (2.6) of [N3], we see that (W, H_0) is the canonical cofree embedding of (W, H) diagonal on the K-basis $\{X_k\}$ and must have

Lemma 2.2.2. $\mathrm{supp}(M_{0j})$ *(*$1 \le j \le n_0$*) are disjoint and* $\prod_{j=1}^{n_0} M_{0j} = \prod_{i=1}^m X_i^{e_i}$. □

Proof of (2.1). Since the ground field K is of characteristic zero, character theory implies that $K[X_1, X_1^{-1}, \ldots, X_m, X_m^{-1}]^{H_i} = K[M_{i'j}, M_{i'j}^{-1} \mid 0 \le i' \le i,\ \forall j]$. Let N be a monomial in $\mathcal{O}(W)^{H_i}$ of $\{X_1, \ldots, X_m\}$. We can choose elements g, h from $K[M_{i'j} \mid 0 \le i' \le i,\ \forall j]$ in such a way that $N = \dfrac{h}{g}$. Suppose that $N \notin K[M_{i'j} \mid i' \le i,\ \forall j]$ and, as $N \in \mathcal{O}(W)^{H_d} = K[M_{pq} \mid \forall p, q]$, let $i_0 > i$ be the smallest number in $\{u \mid N \in K[M_{i'j} \mid i' \le u,\ \forall j]\}$. Express N as a product $N = N' \cdot N''$ of monomials $N' \in K[M_{i'j} \mid i' \le i_0 - 1,\ \forall j]$ and $N'' \in K[M_{i'j} \mid i' \le i_0,\ \forall j]$ of $\{X_p\}$. We may assume that the total degree of N' is large as possible, in this expression $N = N' \cdot N''$. Then N'' must be represented as a monomial of $\{M_{i_0,0}, \ldots, M_{i_0,n_{i_0}}\}$. If N'' is divisible by $\prod_{j=0}^{n_0} M_{i_0j}$ in $K[M_{i_0,j} \mid \forall j]$, by (2.2), since

$$\prod_{j=0}^{n_0} M_{i_0j} \in K[M_{i'j} \mid i' \le i_0 - 1, \forall j],$$

this conflicts with the assumption on the total degree of N'. Thus we may suppose that $N'' = \prod_{j=1}^{n_{i_0}} M_{i_0j}^{c_j}$ for some $c_j \in \mathbf{Z}_0$. Putting $A = K[M_{i'j} \mid i' \le i_0 - 1,\ \forall j]$, we have an A-epimorphism

$$\Delta : A \otimes_K K[T_{i_00}, \ldots, T_{i_0n_{i_0}}] \to K[M_{i'j} \mid i' \le i_0,\ \forall j]$$

defined by $\Delta(T_{i_0j}) = M_{i_0j}$. Clearly, as the elements g, h and $\Theta \circ \Phi(\xi_{i_0})$ belong to A, we have $\mathrm{Ker}(\Delta) = (\prod_{j=0}^{n_{i_0}} T_{i_0j} - \Theta \circ \Phi(\xi_{i_0}))$ (cf. (2.2)) and

$$N'g \cdot \prod_{j=1}^{n_{i_0}} T_{i_0j}^{c_j} - h \in \left(\prod_{j=0}^{n_{i_0}} T_{i_0j} - \Theta \circ \Phi(\xi_{i_0}) \right),$$

which is a contradiction. Consequently we must have

$$\mathcal{O}(W)^{H_i} = K[M_{i'j} \mid 0 \le i' \le i,\ \forall j]$$
$$\cong K\left[\left(\cdots \left(\left(\mathbf{Z}_0^{n_0} \int_{n_1} x_1 \right) \int_{n_2} x_2 \right) \int_{n_3} \cdots \right) \int_{n_i} x_i \right],$$

which is a complete intersection and of codimension i. □

3. A classification of toric hypersuface singularities. Under the notation in (2.2), we suppose that $V//G \cong W//H$ is a singular hypersurface, i.e., $d = 1$, except the definition in (3.4).

Lemma 3.1. $\mathrm{supp}(M_{1i})$*,* $0 \le i \le n_1$*, are disjoint.*

Proof. Suppose that

$$\mathrm{supp}(M_{1i_1}) \cap \mathrm{supp}(M_{1i_2}) \ni X_k$$

for some $0 \le i_1, i_2 \le n_1$, $1 \le k \le m$. Since there exists an index $1 \le u \le n_0$ such that $X_k \in \mathrm{supp}(M_{0u})$ (cf. (2.2.1)), we see that the embedding dimension of $\mathcal{O}(W)^H/(\mathcal{O}(W) \cdot X_k)^H$ is not exceeding $n_0 + n_1 + 1 - 3$, which conflicts with

$$\mathrm{ht}(\mathcal{O}(W) \cdot X_k) = \mathrm{ht}((\mathcal{O}(W) \cdot X_k)^H) = 1. \quad □$$

Put ${}_HW = (\cup_i(\mathrm{supp}(M_{1i})))^\perp$ and

$$W_H = (\{X_1, \ldots, X_m\} \backslash (\cup_i(\mathrm{supp}(M_{1i}))))^\perp,$$

respectively, under the canonical pairing

$$W \times W^\vee \to K.$$

Then $W = {}_HW \oplus W_H$ as H-modules and, by this, we identify $({}_HW)^\vee$ and $(W_H)^\vee$ with the subspaces of $W^\vee$ in a natural way. Put $H_{({}_HW)} = \mathrm{Ker}(H \to GL({}_HW))$ and $H_{(W_H)} = \mathrm{Ker}(H \to GL(W_H))$.

Lemma 3.2. *For $1 \le i \le n_0$, the following conditions are equivalent:*
(1) $\sharp(\mathrm{supp}(M_{0i}) = 1$
(2) $\mathrm{supp}(M_{0i}) \cap (\cup_{k=0}^{n_1}\mathrm{supp}(M_{1k})) = \emptyset$
(3) $\mathrm{supp}(M_{0i}) \not\subseteq (\cup_{j=0}^{n_1}\mathrm{supp}(M_{1j}))]$.
Moreover, we identify $\mathcal{O}({}_HW)^H = \mathcal{O}({}_HW)^{H_{(W_H)}} = K[\{M_{0i} \mid \sharp(\mathrm{supp}(M_{0i})) = 1\}]$, $\mathcal{O}(W_H)^H = O(W_H)^{H_{({}_WH)}} = K[\{M_{0i} \mid \sharp(\mathrm{supp}(M_{0i})) > 1\} \cup \{M_{1j} \mid \forall j\}]$ and $\mathcal{O}(W)^H = \mathcal{O}({}_HW)^{H_{(W_H)}} \otimes_K O(W_H)^{H_{({}_WH)}}$, respectively.

Proof. According to the observation and notation in (2.2), we have

$$K[T_{0i}, T_{1j} \mid \forall i, j]/\left(\prod_{j=0}^{n_1} T_{1j} - \xi_1\right) \cong \mathcal{O}(W)^H,$$

which is induced by Φ. This isomorphism requires that $\Phi(\xi_1)$ is a monomial of $\cup_{j=0}^{n_1}\mathrm{supp}(M_{1j})$. Thus

$$\begin{aligned}(3.3)\quad \mathcal{O}(W)^H &\cong K[M_{0i} \mid \mathrm{supp}(M_{0i}) \\ &\quad \cap(\cup_{j=0}^{n_1}\mathrm{supp}(M_{1j})) = \emptyset] \otimes_K K[M_{0i}, M_{1j} \mid \\ &\quad \mathrm{supp}(M_{0i}) \cap (\cup_{k=0}^{n_1}\mathrm{supp}(M_{1k})) \neq \emptyset, \forall j] \\ &\cong K[M_{0i} \mid \mathrm{supp}(M_{0i}) \not\subseteq \cup_{j=0}^{n_1}\mathrm{supp}(M_{1j})] \\ &\quad \otimes_K K[M_{0i}, M_{1j} \mid \mathrm{supp}(M_{0i}) \\ &\quad \subseteq \cup_{k=0}^{n_1}\mathrm{supp}(M_{1k}), \forall j].\end{aligned}$$

Since $K[M_{0i} \mid \mathrm{supp}(M_{0i}) \cap (\cup_{j=0}^{n_1}\mathrm{supp}(M_{1j})) = \emptyset]$ and

$$K[M_{0i} \mid \mathrm{supp}(M_{0i}) \not\subseteq \cup_{j=0}^{n_1}\mathrm{supp}(M_{1j})]$$

are polynomial rings over K, the ideal $\mathcal{O}(W)^H \cdot M_{0i}$ satisfying

$$\mathrm{supp}(M_{0i}) \cap (\cup_{j=0}^{n_1}\mathrm{supp}(M_{1j})) = \emptyset$$

or $\mathrm{supp}(M_{0i}) \not\subseteq \cup_{j=0}^{n_1}\mathrm{supp}(M_{1j})$ is a prime ideal of height one. By the property of minimal paralleled linear hulls (cf. (1.1) of [N3]), we have $\sharp(\mathrm{supp}(M_{0i})) = 1$ for these M_{0i}'s. Conversely if $\sharp(\mathrm{supp}(M_{0i})) = 1$, then $M_{0i} = X_k^{e_k} \in \mathcal{O}(W)^{I_H(\mathcal{O}(W)\cdot X_k)}$ for some k (cf. (2.2.2) and [N2]), which implies $H|_W = H_{X_k}|_W \times H_{\{X_j|\forall j \neq k\}}|_W$,

$$\mathcal{O}(W)^H = K[X_k^{e_k}] \otimes_K K[X_j \mid \forall j \neq k]^H$$

and $X_k \notin \cup_{j=1}^{n_1}\mathrm{supp}(M_{1j})$ (cf. (2.2)). Here $e_k = \sharp(I_H(\mathcal{O}(W) \cdot X_k)|_W)$ ([N2]). Consequently we have

$$\begin{aligned}H|_{{}_HW} &= H_{(W_H)}|_{{}_HW} \\ &= \prod_{k \in \{k | X_k^{e_k} \in \mathcal{O}(W)^H\}} (I_G(\mathcal{O}(W) \cdot X_k)|_{{}_HW}),\end{aligned}$$

which implies $H|_W = H_{(W_H)}|_W \times H_{({}_HW)}|_W$. The last assertion follows from these remarks. □

Definition 3.4. Let Γ be a subset of $\{X_1^{e_1}, \dots, X_m^{e_m}\}$ and $p, q \in \mathbf{Z}_0$. Recall $e_i = \sharp(I_H(\mathcal{O}(W) \cdot X_i)|_W)$ (cf. [N2]). We say that a finite set Λ of monomials of $\{X_1^{e_1}, \dots, X_m^{e_m}\}$ has a (p, q)-matrix structure of weight $(a_1, \dots, a_p) \in \mathbf{N}^p$ on Γ, if $\sharp(\Gamma) = p \cdot q$ and

$$\Lambda = \left\{ \prod_{v=1}^{q} Y_{iv}, \prod_{u=1}^{p} Y_{uj}^{a_u} \,\middle|\, 1 \le i \le p, 1 \le j \le q \right\},$$

where $\{Y_{ij} \mid 1 \le i \le p, 1 \le j \le q\} = \Gamma$ whose elements are indexed doubly.

Lemma 3.5. *The set*

$$\{M_{0i} \mid \sharp(\mathrm{supp}(M_{0i})) > 1\} \cup \{M_{1j} \mid 0 \le j \le n_1\}$$

has a (p, q)-matrix structure of weight $(a_1, \dots, a_p)$ on $\Gamma = \{X_k^{e_k} \mid X_k \in \widetilde{\Gamma}\}$ for some $p, q \in \mathbf{N}$ such that $\sharp(\Gamma) = p \times q$ and $(a_1, \dots, a_p) \in \mathbf{N}^p$, where $\widetilde{\Gamma} = \cup_{j=0}^{n_1}\mathrm{supp}(M_{1j})$. Moreover $q = n_1 + 1$ and the condition $p \ge 2$ or $a_1 \ge 2$ holds.

Proof. Let i be any index such that $\sharp(\mathrm{supp}(M_{0i})) > 1$. If there exists $0 \le j \le n_1$ such that $\mathrm{supp}(M_{0i}) \cap \mathrm{supp}(M_{1j}) = \emptyset$, then we see

$$\mathrm{supp}\left(\prod_{k \neq j} M_{1k}\right) \supseteq \mathrm{supp}(M_{0i}),$$

which implies $\prod_{k \neq j} T_{1k} - T_{0i} \cdot \gamma \in \mathrm{Ker}(\Phi)$ for some $\gamma \in K[T_{0u}, T_{1v} \mid \forall u, v]$. This conflicts with (2.2). Suppose that $\mathrm{supp}(M_{0i}) \cap \mathrm{supp}(M_{1j})$ contains X_1 and X_2. Both $\mathcal{O}(W)^H/\mathcal{O}(W)\cdot X_k \cap \mathcal{O}(W)^H$, $k = 1, 2$, are of embedding dimension at most $n_0 + n_1 + 1 - 2$. Hence, as $\mathrm{ht}(\mathcal{O}(W) \cdot X_k \cap \mathcal{O}(W)^H) = 1$, we must have

$$\begin{aligned}\mathcal{O}(W) \cdot X_1 \cap \mathcal{O}(W)^H &= (M_{0i}, M_{1j}) \\ &= \mathcal{O}(W) \cdot X_2 \cap \mathcal{O}(W)^H.\end{aligned}$$

This also conflicts with the property of the minimal paralleled linear hull (cf. (1.1) of [N3]). Consequently, by (3.1) and (3.3), we index the set Γ doubly as follows: Let $\widetilde{\Gamma}$ be the set

$$\{Z_{uv} \mid 1 \le u \le p, 1 \le v \le q\}$$

defined by

$$\mathrm{supp}(M_{0t_i}) \cap \mathrm{supp}(M_{1j}) = \{Z_{ij+1}\}$$

for $1 \leq i \leq p$ and $0 \leq j \leq n_1$, where

$$\{M_{0t_i} \mid 1 \leq i \leq p\} = \{M_{0k} \mid \sharp(\mathrm{supp}(M_{0k})) > 1\},$$

$t_1 < t_2 < \cdots < t_p$, and $q = n_1 + 1$. For $Z_{uv} = X_k$, let e_{uv} denote e_k, and put $Y_{uv} = Z_{uv}^{e_{uv}}$ ($1 \leq u \leq p$, $1 \leq v \leq q$). Then $M_{0t_i} = \prod_{v=1}^{q} Y_{iv}$ (cf. (2.2.2)) and $M_{1j-1} = \prod_{u=1}^{p} Y_{uj}^{c_{uj}}$ for some $c_{uj} \in \mathbf{N}$. By (3.2), (3.3) and the observation in (2.2), there is a unique relation

$$\prod_{j=0}^{n_1} M_{1j} = \prod_{i=1}^{p} M_{0t_i}^{d_i}$$

with certain $d_i \in \mathbf{N}$. Therefore we see that c_{uj}'s are independent of j, and so denoted it by a_u. The almost all assertions follow from these facts and the last one is a consequence of non-regularity of $\mathcal{O}(W)^H$. □

Definition 3.6. For any weight $\chi \in \mathcal{W}(W, H)$, let $e(\chi)$ be the number $\sharp(\chi(\mathcal{W}(W/W_\chi, H)^\perp))$, where the orthogonal set is related to the pairing

$$H \times \mathfrak{X}(H) \to K^\times$$

($e(\chi) = \infty$, if $\chi(\mathcal{W}(W/W_\chi, H)^\perp)$ is of infinite order). In the case where $\dim W_\chi = 1$, we have $e(\chi) = \sharp(I_H(\mathcal{O}(W) \cdot (W^\vee)_{-\chi})|_W) < \infty$ (cf. [N3]).

For a finite subset Ξ of $\mathfrak{X}(H)$ such that $\mathbf{Z}_0 \cdot \Xi = \mathbf{Z} \cdot \Xi$, define the canonical epimorphism $\nu : \oplus_{\lambda \in \Xi} \mathbf{Z} \cdot \lambda \to \mathbf{Z} \cdot \Xi$ such that $\nu(\lambda) = \lambda$. We say that an element

$$F \in \sum_{\lambda \in \Xi} \mathbf{Z}_0 \cdot \lambda \subseteq \oplus_{\lambda \in \Xi} \mathbf{Z} \cdot \lambda$$

is *a non-negative relation of* Ξ, if $\lambda(F) = 0$, and, for a convenience sake, we denote the relation F by $F = 0$. For

$$F_i \in \sum_{\lambda \in \Xi} \mathbf{Z}_0 \cdot \lambda \subseteq \oplus_{\lambda \in \Xi} \mathbf{Z} \cdot \lambda \ (1 \leq i \leq l),$$

the set Ξ has the *basic relations* $F_1 = 0, \ldots, F_l = 0$, if any non-negative relation $F = 0$ satisfies that

$$F \in \sum_{i=1}^{l} \mathbf{Z}_0 \cdot F_i \subseteq \sum_{\lambda \in \Xi} \mathbf{Z}_0 \cdot \lambda \subseteq \oplus_{\lambda \in \Xi} \mathbf{Z} \cdot \lambda.$$

Theorem 3.7. *Let (W, w) be a minimal paralleled linear hull of (V, G) and put $H = G_w$. Then $V//G$ is a singular hypersurface if and only if the following conditions on $\mathcal{W}(W, H)$ are satisfied:*

(1) *$\sharp(\mathcal{W}(W, H) \backslash \{0\}) = \dim W/W^H = p \cdot q + s$ for some p, $q \in \mathbf{N}$ and $s \in \mathbf{Z}_0$.*

(2) *We can express $\mathcal{W}(W, H) \backslash \{0\} = \Psi \sqcup \{\chi_{ij} \mid 1 \leq i \leq p, 1 \leq j \leq q\}$ (disjoint sum) with $\sharp(\Psi) = s$ such that $\infty > \sharp(\psi(H)) = e(\psi) > 1$ for any $\psi \in \Psi$ and $\{e(\chi_{ij}) \cdot \chi_{ij} \mid 1 \leq i \leq p, 1 \leq j \leq q\}$ has the non-negative basic relations*

$$\begin{cases} \sum_{j=1}^{q} e(\chi_{ij}) \cdot \chi_{ij} = 0 & (1 \leq i \leq p) \\ \sum_{i=1}^{p} a_i e(\chi_{ij}) \cdot \chi_{ij} = 0 & (1 \leq j \leq q) \end{cases}$$

for some $(a_1, \ldots, a_p) \in \mathbf{N}^p$. Here $q > 1$ holds and, if $p = 1$, the inequality $a_1 > 1$ is satisfied.

Combining the next lemma with (3.2) and (3.5), we will give a paraphrase in terms of characters, which completes the proof of this theorem.

Lemma 3.8. *Let Γ be a subset of $\{X_1^{e_1}, \ldots, X_m^{e_m}\}$ for any $e_i \in \mathbf{N}$. Let Λ be a finite set of monomials of $\{X_1^{e_1}, \ldots, X_m^{e_m}\}$ having a (p, q)-matrix structure of weight $(a_1, \ldots, a_p) \in \mathbf{N}^p$ on Γ for any p, $q \in \mathbf{Z}_0$ and $(a_1, \ldots, a_p) \in \mathbf{N}^p$. Then there exists a unique closed subgroup L of $GL(W)$ such that $\mathcal{O}(W)^L = K[(\{X_1^{e_1}, \ldots, X_m^{e_m}\} \backslash \Gamma) \cup \Lambda]$.*

Proof. Suppose that

$$\Lambda = \left\{ \prod_{v=1}^{q} Y_{iv}, \prod_{u=1}^{p} Y_{uj}^{a_u} \,\middle|\, 1 \leq i \leq p, 1 \leq j \leq q \right\},$$

where $\{Y_{ij} \mid 1 \leq i \leq p, 1 \leq j \leq q\}$ denotes Γ. Put $\widetilde{\Lambda} = \{(\prod_{v=1}^{q} Y_{iv})^{a_i}, \prod_{u=1}^{p} Y_{uj}^{a_u} \mid 1 \leq i \leq p, 1 \leq j \leq q\}$. Let S denote the stabilizer of the diagonal group $D_{\{Y_{ij}^{a_i}\}}(\bigoplus_{i,j} K \cdot Y_{ij}^{a_i})$ at the set $\widetilde{\Lambda}$ under the natural action of $D_{\{Y_{ij}^{a_i}\}}(\bigoplus_{i,j} K \cdot Y_{ij}^{a_i})$ on $K[Y_{ij}^{a_i} \mid \forall i, j]$. Then $\dim S = p \cdot q - p - q + 1$ and $K[Y_{ij} \mid 1 \leq i \leq p, 1 \leq j \leq q]^{\rho^{-1}(S)} = K[\widetilde{\Lambda}]$, where

$$\rho : D_{\{Y_{ij}\}}\left(\bigoplus_{i,j} K \cdot Y_{ij}\right) \to D_{\{Y_{ij}^{a_i}\}}\left(\bigoplus_{i,j} K \cdot Y_{ij}^{a_i}\right)$$

is the canonical epimorphism. From this, we infer that the kernel of the K-homomorphism

$$\varrho : K[U_{01}, \ldots, U_{0p}, U_{11}, \ldots, U_{1q}] \to K[\Lambda]$$

from the $(p+q)$-dimensional polynomial algebra over K sending

$$U_{0i} \mapsto \prod_{j=1}^{p} Y_{ij}, U_{1j} \mapsto \prod_{i=1}^{q} Y_{ij}^{a_i}$$

is principal and contains the irreducible polynomial

$$\prod_{j=1}^{q} U_{1j} - \gamma$$

for some monomial $\gamma \in K[U_{01}, \ldots, U_{0p}]$. As

$$K[Y_{ij}, Y_{ij}^{-1} \mid i, j]^{(D_{\{Y_{ij}^{a_i}\}}(\oplus_{\forall i,j} K \cdot Y_{ij}^{a_i}))_\Lambda} = K[\Lambda \cup \{x^{-1} \mid \forall x \in \Lambda\}],$$

as in the proof of (3.1), we similarly have

$$(3.8.1)\quad K[Y_{ij} \mid i,j]^{(D_{\{Y_{ij}^{a_i}\}}(\bigoplus_{i,j} K\cdot Y_{ij}^{a_i}))_\Lambda} = K[\Lambda].$$

Here $(\bullet)_\Lambda$ denotes the stabilizer of $\bullet$ at Λ. Since $K[X_1^{e_1},\dots,X_m^{e_m}] \cong K[\{X_1^{e_1},\dots,X_m^{e_m}\}\backslash\Gamma] \otimes_K K[\Gamma]$ is a ring of invariants of a finite diagonal group generated by pseudo-reflections in GL(W) (cf. [S]), by (3.8.1), we have a closed subgroup L of $GL(W)$ satisfying

$$\mathcal{O}(W)^L = K[(\{X_1^{e_1},\dots,X_m^{e_m}\}\backslash\Gamma)\cup\Lambda].$$

The uniqueness of L follows similarly from stability of the action as in the proof of (3.3) of [N3]. □

Proof of (3.7). To prove (3.7), we may assume that $W^H = \{0\}$. By (3.5) and (3.8), $W//H$ is a hypersurface if and only if there exists a subset Γ of $\{X_1^{e_1},\dots,X_m^{e_m}\}$ and a finite set of monomials Λ of $\{X_1^{e_1},\dots,X_m^{e_m}\}$ having a (p,q)-matrix structure of weight $(a_1,\dots,a_p) \in \mathbf{N}^p$ on Γ such that any monomial in $\mathcal{O}(W)^H$ is represented as a monomial of the members of $\Lambda \cup (\{X_1^{e_1},\dots,X_m^{e_m}\}\backslash\Gamma)$. Here $e_i = \sharp(I_H(\mathcal{O}(W)\cdot X_i)|_W)$ (cf. [N2]).

Suppose that the condition of the "only if" part of the statement as above holds. Let Ψ denote the set of $\psi \in \mathcal{W}(W,H)$ satisfying the both conditions $(W^\vee)_{-\psi} \ni X_i$ and $X_i^{e_i} \in \{X_1^{e_1},\dots,X_m^{e_m}\}\backslash\Gamma$. Then, as $X_i^{e_i} \in \mathcal{O}(W)^H$ for $X_i^{e_i} \notin \Gamma$, we have $\sharp(\psi(H)) = e(\psi) = e_i$ and $(W^\vee)_{-\psi} = K\cdot X_i$ for the $\psi \in \Psi$ such that $(W^\vee)_{-\psi} \ni X_i$. Moreover, we see $\sharp(\Psi) = m - \sharp(\Gamma)$. Let

$$\theta : \{1,\dots,p\}\times\{1,\dots,q\} \to \{1,\dots,m\}$$

be a map defined by $X_{\theta(i,j)}^{e_{\theta(i,j)}} = Y_{ij}$. For $1 \le i \le p$ and $1 \le j \le q$, let $\chi_{ij} \in \mathcal{X}(H)$ denote the character such that $(W^\vee)_{-\chi_{ij}} \ni X_{\theta(ij)}$. Then $(W^\vee)_{-\chi_{ij}} = K\cdot X_{\theta(ij)}$, $e(\chi_{ij}) = e_{\theta(i,j)}$ and $\dim W = p\cdot q + \sharp(\Psi)$. Since the K-algebra $\mathcal{O}(W)^H$ is generated by $\Lambda \cup (\{X_1^{e_1},\dots,X_m^{e_m}\}\backslash\Gamma)$, the basic non-negative relations of $\{e(\chi_{ij})\cdot\chi_{ij}\}$ are identified with the relations in (2) of (3.7).

Conversely, if the conditions (1) and (2) in (3.7) hold, then we show that $W//H$ is a hypersurface by following reverse of the argument mentioned above. It should be noted that the non-regularity of $W//H$ is related to the last condition in (2) in (3.7). □

3.9. Examples for (3.7). In order to explain the content of (3.7), we give the following examples. For a convenience sake, suppose that $\mathcal{R}_W(H)|_W = \{1\}$ in the both examples.

Example 3.9.1. Under the same circumstances as in (2.7.1), $V//G$ is a singular hypersurface if and only if the conditions (1) and (2) in (3.7) in the case where $s = 0$, $p = 1$ $(a_1 > 1)$ and $e(\chi_{1j}) = 1$ hold. In other word, this is equivalent to the condition that there exists a direct sum $W = W^H \oplus W_H$ of KH-modules such that $H|_{W_H}$ can be represented as a subgroup $< \mathrm{diag}[\zeta_a, \zeta_a^{-1}, 1,\dots,1], \mathrm{diag}[\zeta_a, 1, \zeta_a^{-1}, 1,\dots,1],\dots,\mathrm{diag}[\zeta_a, 1,\dots,1,\zeta_a^{-1}] >$ of $GL(W_H)$ on some K-basis, where $\zeta_a \in K$ is a fixed primitive a-th root of unity for some $1 < a = a_1 \in \mathbf{N}$.

Example 3.9.2. Under the same circumstances as in (2.7.2), we suppose that $W^H = \{0\}$. Then $V//G$ is a singular hypersurface if and only if $m = 4$ and there exists a bijection

$$\theta : \{(1,1),(1,2),(2,1),(2,2)\} \to \{1,2,3,4\}$$

such that the relations of $\{\xi_1|_H,\dots,\xi_m|_H\}$ are only

$$\begin{cases} \xi_{\theta((i,1))}|_H + \xi_{\theta((i,2))}|_H = 0 & (i = 1,2) \\ a_1\cdot\xi_{\theta((1,j))}|_H + a_2\cdot\xi_{\theta((2,j))}|_H = 0 & (j = 1,2) \end{cases}$$

for some $(a_1,a_2) \in \mathbf{N}^2$. The character $\xi_{\theta((i,j))}|_H$ is regarded as χ_{ij} in (3.7).

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