Kummer sandwich theorem of certain elliptic K3 surfaces

By Tetsuji SHIODA

Department of Mathematics, Rikkyo University, 3–34–1 Nishi-Ikebukuro,
Toshima-ku, Tokyo 171–8501, Japan

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Abstract: It is shown that any elliptic K3 surface with a section and with two II^*-fibres is sandwiched by a Kummer surface in a very precise way.

Key words: K3 surface; elliptic surface; singular fibres; Kummer surface; isogeny.

1. Introduction. The aim of this note is to prove the following result which may be called the "Kummer sandwich theorem" of any elliptic K3 surface (with a section) with two II^*-fibres by a Kummer surface.

Theorem 1.1. *Suppose X is any elliptic K3 surface with a section and with two II^*-fibres. Then there exist a unique Kummer surface $S = \mathrm{Km}(C_1 \times C_2)$ of the product of two elliptic curves C_1, C_2 and two commuting symplectic involutions $\sigma, \tau \in \mathrm{Aut}(S)$ such that (i) the quotient surface $S/\langle\sigma\rangle$ is birational to X, and (ii) the quotient surface $S/\langle\sigma,\tau\rangle$ is birational to S itself. In particular, S dominates and is dominated by X, by the rational maps of degree two:*

$$(1.1) \qquad \phi : S \to X, \quad \psi : X \to S.$$

The last assertion (1.1) expressing the "isogenous" relationship of X, S has been shown in [1] and [2] in the case of complex singular K3 surfaces (see the remark in §4). Theorem 1.1 is a refined generalization of this fact. The proof, to be given in §4, is purely algebraic and works for any algebraically closed base field k of characteristic $\neq 2, 3$. It makes use of a recent result (see §3) on the explicit defining equations of elliptic fibrations on a Kummer surface [5].

A few words are in order for the motivation to study those special K3 surfaces treated here. It is wellknown that in the proof of the Torelli theorem (injectivity of the period map) by Piateckii-Shapiro-Shafarevich [8] a special role is played by the Kummer surfaces which are dense in the moduli space of polarized K3 surfaces. The elliptic K3 surfaces with two II^*-fibres are first constructed in Inose-Shioda

[2] (corresponding to the above ψ) and used for proving the surjectivity of the period map in the case of singular K3 surfaces, which gives the complete classification of such surfaces. This construction is extended by Morrison [6] in a useful way. Another construction given by Inose [1] (corresponding to the above ϕ) has recently been reconsidered by Kuwata [4] and others (cf. [9, 10]), one reason being that it leads to elliptic K3 surfaces with high Mordell-Weil rank. Thus these K3 surfaces, very special as they may be from the moduli point of view, have various rich properties, which we think are still worth studying today. Along the same line, an approach to the notion of isogeny for K3 surfaces has been proposed in the case of singular K3 surfaces in [1, 2]. The above theorem illustrates that an elliptic K3 surface X with two II^*-fibres is "isogenous" to a Kummer surface S in a very concrete sense. By the way, it is an open question to decide whether or not the existence of a dominant rational map of K3 surfaces $X \to Y$ is symmetric with respect to X, Y as in the case of abelian varieties.

Notation. For the singular fibres of an elliptic surface, we follow Kodaira's notation [3]; also we use freely the results on singular fibres from [3, 11]. For abbreviation, let us introduce the following classes of algebraic K3 surfaces (over a base field k) up to isomorphisms:

(KmP): the Kummer surfaces $\mathrm{Km}(C_1 \times C_2)$ of the product of two elliptic curves C_1, C_2.

(II^{*2}): the elliptic K3 surfaces with a section and with two singular fibres of type II^* (at $0, \infty \in \mathbf{P}^1$) plus some other singular fibres.

(II^*I^{*2}): the elliptic K3 surfaces with a section and with one singular fibre of type II^* (at $\infty \in \mathbf{P}^1$) and two of types $I_{b_1}^*, I_{b_2}^*$ (or I_0^*, IV^*) and possibly

some others.

$\mathbf{P}_u^1$: the projective line with the inhomogeneous coordinate u (the u-line).

$L[n]$: the lattice L with the pairing multiplied by n.

T_X: the lattice of transcendental cycles on a surface X (over $k = \mathbf{C}$), which carries a natural Hodge structure (as a sub-Hodge structure of $H^2(X, \mathbf{Z})$).

2. The defining equation for $X \in (II^{*2})$. Let X be an elliptic K3 surface over the T-line $\mathbf{P}_T^1$ belonging to (II^{*2}).

Proposition 2.1. *Every $X \in (II^{*2})$ has the defining Weierstrass equation*

$$(2.1) \qquad X_{\alpha,\beta} : \ y^2 = x^3 - 3\alpha x + (T + \frac{1}{T} - 2\beta)$$

for some constants α, β. The pair (α^3, β^2) is uniquely determined by X.

Proof. We can write the Weierstrass equation of an elliptic K3 surface (with a section) X as

$$y^2 = x^3 + Ax + B$$

with polynomial coefficients $A, B \in k[T]$, $\deg A \leq 8$, $\deg B \leq 12$. By [3] or [11], a necessary condition for this to have a fibre of type II^* at $T = 0$ is that $A = T^4 A_1$, $B = T^5 B_1$ for some $A_1, B_1 \in k[T]$ and $B_1(0) \neq 0$. Considering the same condition at $T = \infty$, we have $A_1 \in k$ and $\deg B_1 = 2$. Then by replacing T by a constant multiple, we can assume $B_1 = b_0(T^2 + 1) + b_1 T$. Next, by replacing x, y by suitable constant multiples, we can make $b_0 = 1$. Thus we have

$$y^2 = x^3 + A_1 T^4 x + T^5(T^2 + 1 + b_1 T).$$

Dividing the both sides by T^6 and letting $\alpha = -A_1/3$ and $\beta = -b_1/2$, we obtain the equation in the desired form (2.1). Conversely it is easy to check that this equation defines an elliptic K3 surface with two II^*-fibres.

After choosing the coordinates T, x, y as above, the remaining freedom for them is

$$T \to T^{\pm 1}, \quad x \to \omega x, \quad y \to \pm y \quad (\omega^3 = 1).$$

This implies the uniqueness of α, β as asserted. $\quad\square$

3. Elliptic pencils on a Kummer surface. Let $S = \mathrm{Km}(C_1 \times C_2)$ be the Kummer surface of the product of two elliptic curves C_1, C_2. Let j_1, j_2

denote their j-invariants. (The j-invariant is classically normalized so that $j = 1$ for the elliptic curve $y^2 = x^3 - x$ instead of $j = 1728$.) In the sequel, we write such an S as

$$S = S_{j_1, j_2}.$$

Proposition 3.1. *The Kummer surface $S = S_{j_1, j_2}$ admits an elliptic fibration $f : S \to \mathbf{P}_t^1$ which has two singular fibres of type IV^* (Inose's pencil). Its defining equation is given by*

$$(3.1) \qquad y^2 = x^3 - 3\alpha x + (t^2 + \frac{1}{t^2} - 2\beta)$$

where

$$(3.2) \qquad \alpha = \sqrt[3]{j_1 j_2}, \ \beta = \sqrt{(1 - j_1)(1 - j_2)}$$

with the choice of the cube root or square root being arbitrary. As a function on the surface S, the elliptic parameter t is equal, up to a constant, to the ratio y_2/y_1 of the y-coordinates of Weierstrass equations of C_1, C_2.

Proof. For the proof, see [1, 9, 10] or [5, §2]. $\quad\square$

Proposition 3.2. *The Kummer surface $S = S_{j_1, j_2}$ admits another elliptic fibration $f' : S \to \mathbf{P}_u^1$ belonging to the class $(II^* I^{*2})$, i.e. it has (at least) three singular fibres $II^*, I_{b_1}^*, I_{b_2}^*$, or II^*, I_0^*, IV^*. If we normalize the position of these fibres (in this order) at the 3 points $u = \infty$, $u = \pm 2$ of the base curve $\mathbf{P}_u^1$, the defining equation is given by*

$$(3.3) \quad y^2 = x^3 - 3\alpha(u^2 - 4)^2 x + (u - 2\beta)(u^2 - 4)^3$$

where α, β are determined by (3.2).

Proof. For the proof, see [2] for the first assertion. The second follows from [5, §5] by a simple coordinate change. $\quad\square$

Remark. The elliptic fibrations f (or f') on S is known to be unique up to automorphisms of S, at least if $k = \mathbf{C}$ and C_1, C_2 are not isogenous to each other. See Oguiso [7].

4. Proof of Theorem 1.1. Take any K3 surface X in the class (II^{*2}). By Propositions 2.1, we can assume that $X = X_{\alpha,\beta}$, i.e. it is defined by the equation (2.1) for some constants $\alpha, \beta \in k$. Choose two elements $j_1, j_2 \in k$ such that

$$(4.1) \qquad j_1 j_2 = \alpha^3, \quad j_1 + j_2 = 1 + \alpha^3 - \beta^2.$$

The (unordered) pair $\{j_1, j_2\}$ is uniquely determined by X. Let C_1 (or C_2) be the elliptic curve with the

j-invariant j_1 (or j_2), and let $S = S_{j_1,j_2} = \text{Km}(C_1 \times C_2)$.

Then, by comparing Propositions 2.1 and 3.1, we have a rational map of degree two from S to X defined by

$$\phi : S \to X, \quad (x,y,t) \mapsto (x,y,T), \quad T = t^2.$$

Next we consider the group $G = \langle \sigma, \tau \rangle$ of symplectic automorphisms of S generated by the involutions:

$$\sigma : (x,y,t) \mapsto (x,y,-t), \quad \tau : (x,y,t) \mapsto (x,-y,1/t).$$

The above map ϕ is birationally equivalent to the quotient map $S \to S/\langle\sigma\rangle$. Hence the quotient S/G is birational to $Y := X/\langle\bar\tau\rangle$, where $\bar\tau$ is the involution of X induced by τ:

$$\bar\tau : (x,y,T) \mapsto (x,-y,1/T).$$

The invariant subfield of $k(x,y,T)$ under this involution is generated by the 3 elements x, $s := T + \frac{1}{T}$, $z := y(T - \frac{1}{T})$ which satisfy the relation

$$z^2 = y^2(T - \frac{1}{T})^2 = (x^3 - 3\alpha x + (s - 2\beta))(s^2 - 4).$$

Hence, by letting

$$\xi := x(s^2 - 4) = x(T - \frac{1}{T})^2, \ \eta := z(s^2 - 4) = y(T - \frac{1}{T})^3,$$

the function field of the quotient Y is equal to $k(Y) = k(s, \xi, \eta)$ with the relation

$$\eta^2 = \xi^3 - 3\alpha\xi(s^2 - 4)^2 + (s - 2\beta)(s^2 - 4)^3.$$

Now comparing the last equation with (3.3) in Proposition 3.2, we see that Y is birationally equivalent to S. Therefore the map

$$\psi : \quad (x,y,T) \mapsto (\xi,\eta,s)$$

defines a rational map of degree two from X to S, which is the quotient map of X by the involution $\bar\tau$.

This completes the proof of Theorem 1.1.   □

Remark. For comparison, we outline the (transcendental) proof of the claim (1.1) in [1, 2] ($k = \mathbf{C}$). Take any Kummer surface $S \in (KmP)$. (i) Via the double cover construction $\psi : X \to S$, we obtain $X \in (II^{*2})$ such that $T_S \cong T_X[2]$ for the lattices of transcendental cycles. (ii) On the other hand, via the double quotient construction $\phi : S \to Y$ using

the Inose's pencil on S, we obtain $Y \in (II^{*2})$ such that $T_S \cong T_Y[2]$. (iii)Thus we have $X, Y \in (II^{*2})$ and a rational map $X \to S \to Y$ of degree 4, such that $T_X \cong T_Y$. Then the Torelli theorem implies that X and Y are isomorphic K3 surfaces. (In [1] and [2], the case of "singular" K3 surfaces is treated.) Thus we have the rational map $S \to Y \cong X \to S$ of degree 4 as in (1.1).

At any rate, this argument has predicted the explicit defining equation of the elliptic fibration on $S \in (KmP)$ stated in Prop.3.2 (verified in [5]), and suggested the Kummer sandwich theorem.

5. Some consequences. In the above proof, a bijective correspondence is given between the (unordered) pairs $\{j_1, j_2\}$ and the pairs (α^3, β^2) related by (3.3) or (4.1), i.e.

$$j_1 j_2 = \alpha^3, \quad j_1 + j_2 = 1 + \alpha^3 - \beta^2.$$

Then the following is immediate.

Proposition 5.1. *The correspondence* $S_{j_1,j_2} \mapsto X_{\alpha,\beta}$ *gives a bijection from* (KmP) *to* (II^{*2}).

Proposition 5.2. *The correspondence* $S = S_{j_1,j_2} \mapsto (S, f')$ *defines a bijection from* (KmP) *to* $(II^* I^{*2})$.

Proof. Take any $Z \in (II^* I^{*2})$. Taking the base change, we obtain $X \in (II^{*2})$ with an involution σ such that $X/\langle\sigma\rangle \sim Z$. It is clear that Z is the unique quotient of X belonging to $II^* I^{*2}$. It follows from Prop.5.1 that $Z \cong S_{j_1,j_2}$ for some j_1, j_2. This proves that the correspondence (KmP) to (II^{*2}) is surjective. The injectivity is obvious in this situation.   □

Corollary 5.3. *Every* $Z \in (II^* I^{*2})$ *is a Kummer surface.*

In the complex case $k = \mathbf{C}$, the above Corollary was stated without proof in [2, §2].

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