Remarks on global behavior of solutions to nonlinear Schrödinger equations

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Abstract: We consider the initial value problem for systems of cubic nonlinear Schrödinger equations in one space dimension with small initial data. We present a structural condition on the nonlinearity under which the solution exists globally in time and behaves like a free solution at infinity. This condition corresponds to an NLS version of the *null condition*.

Key words: Nonlinear Schrödinger equations; global existence; asymptotic behavior; null condition.

1. Introduction. We consider the nonlinear Schrödinger equations (NLS) of the following type:

$$\left(i\partial_t + \frac{1}{2}\partial_x^2\right) u_k = F_k(u, \partial_x u) \tag{1}$$

in $(t,x) \in [0,\infty) \times \mathbf{R}$, $k = 1, \cdots, m$, with the initial condition

$$u_k(0,x) = \varphi_k(x), \tag{2}$$

where $u = (u_k(t,x))_{1\le k\le m}$ is a $\mathbf{C}^m$-valued unknown function, $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $i = \sqrt{-1}$ and φ_k are given smooth functions ($k = 1, \cdots, m$). The nonlinear terms F_k are supposed to depend on $(u, \partial_x u)$ smoothly in the real sense and vanish at cubic order as $(u, \partial_x u) \to (0,0)$. We also assume that the nonlinear terms are gauge invariant, i.e., $F_k(e^{i\theta}v, e^{i\theta}q) = e^{i\theta}F_k(v,q)$ for all $v, q \in \mathbf{C}^m$, $\theta \in \mathbf{R}$, $k = 1, \cdots, m$.

This article is devoted to the study on large time asymptotic behavior of solutions to (1)–(2). From the viewpoint of large time behavior, it is well known that one dimensional, cubic nonlinear case is a delicate case. The difficulty is not just a technical one because we can not expect sufficient decay of the nonlinear terms in general. In other words, influence of the nonlinearity is so strong that the solutions do not behave like a solution of the free Schrödinger equation (which we call a "free solution" in what follows) even if the initial data are sufficiently small and smooth. (This should be compared with the higher dimensional case. See e.g., [Chi1].) So our main interest is to clarify how the cubic nonlinearity affects the global behavior of the solution. Although there are a large amount of results on this issue (see e.g., [HN1, HN2, HNU, HO, KT, O1, O2, Sh, T] and the references therein), we would have to say that the principle is still not clearly understood particularly in the case of NLS systems. For the nonlinear wave equations (NLW), the nonlinear effect is studied extensively in connection with the *null condition* and it allows us to understand, at least on the formal level, how the nonlinearity affects the large time behavior of the solution with small initial data. We refer the readers to [A1, Chr, H, J, K] for the basic references (see also [A2, A3, L, LR1, LR2] etc. for the recent progress on NLW in connection with the *weak null condition*). There have been several attempts to establish the NLS version of the null condition previously (for instance, [KT, T] etc.). However, it seems that they have not been as successful as the NLW case.

The aim of this paper is to introduce the null condition for NLS, which is quite analogous to that for NLW. We will show that the small amplitude solution exists globally in time and behaves like a free solution at infinity under this condition. Our null condition is fairly sharp as a condition for the solution being asymptotically free although it is not optimal from the viewpoint of small data global existence. To close the introduction, we must mention that our approach is not well-suited for the quadratic nonlinear case (even in two or three space dimensions) because it depends heavily on the gauge in-

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variance property.

2. The Null Condition for NLS. Let us first introduce a notation: Let $F^{(\mathrm{c})}$ denote the cubic homogeneous part of a cubic nonlinear term F so that $F^{(\mathrm{c})}(u,\partial_x u)$ gives the main contribution and $F(u,\partial_x u)-F^{(\mathrm{c})}(u,\partial_x u)$ is regarded as a higher order remainder term when one considers small amplitude solutions. With this notation, we define the null condition as follows:

Definition 1. We say that the system (1) satisfies the *null condition* if $F_k^{(\mathrm{c})}(v,i\xi v)=0$ for all $\xi\in\mathbf{R}$, $v\in\mathbf{C}^m$, $k=1,\cdots,m$.

Remark 1. Our null condition covers the "null gauge condition of order 3" in the sense of Y.Tsutsumi as a special case (see Definition 1.1 of [T]).

To state our main result, we introduce the weighted Sobolev space $H^{s,\sigma}=\{f\in L^2;\|f\|_{H^{s,\sigma}}<\infty\}$, where $\|f\|_{H^{s,\sigma}}=\|(1+x^2)^{\sigma/2}(1-\partial_x^2)^{s/2}f\|_{L^2}$. Then we have the following

Theorem 1. *Let $s\ge 5$. Suppose that the system (1) satisfies the null condition. Then there exists a constant $\varepsilon>0$ such that (1)–(2) admits a unique global solution*

$$u\in\bigcap_{j=0}^{s}C([0,\infty);H^{s+1-j,j}),$$

provided that $\varphi=(\varphi_k)_{1\le k\le m}\in\cap_{j=0}^{s}H^{s+1-j,j}$ and $\sum_{j+k\le s}\|x^j\partial_x^k\varphi\|_{L^2}\le\varepsilon$. Moreover, $u(t)$ behaves like a free solution as $t\to\infty$ in the following sense: There exists $(\varphi_k^+)_{1\le k\le m}\in L^2$ such that

$$\lim_{t\to\infty}\sum_{k=1}^{m}\|u_k(t,\cdot)-e^{it\partial_x^2/2}\varphi_k^+\|_{L^2}=0.$$

We shall give an outline of the proof of Theorem 1. Put $J=x+it\partial_x$. It is known that this operator has good compatibility with the Schrödinger operator, for instance, we have $\left[i\partial_t+\frac{1}{2}\partial_x^2,J\right]=0$, $[\partial_x,J]=1$, and so on, where $[\cdot,\cdot]$ denotes the commutator. Another important property is that the inequality

$$\|u(t,\cdot)\|_{L^\infty}\le\frac{C}{\sqrt{1+t}}\sum_{j+k\le 1}\|J^j\partial_x^k u(t,\cdot)\|_{L^2}\tag{3}$$

is valid with some positive constant C (see e.g. Lemma 3.1 of [T] for the details). Also we introduce the notation

$$\frac{\partial}{\partial q_k}=\frac{1}{2}\left(\frac{\partial}{\partial X_k}-i\frac{\partial}{\partial Y_k}\right),\ \frac{\partial}{\partial\overline{q_k}}=\frac{1}{2}\left(\frac{\partial}{\partial X_k}+i\frac{\partial}{\partial Y_k}\right)$$

with $\operatorname{Re}q_k=X_k$, $\operatorname{Im}q_k=Y_k$ (here we associate the variable q_k with $\partial_x u_k$). The key observation is the following one: First we note that

$$i\frac{x}{t}=\partial_x+\frac{i}{t}J.$$

Then, since the null condition yields

$$F_j^{(\mathrm{c})}\Big(u,\partial_x u+\frac{i}{t}Ju\Big)=0,$$

it follows that

$$\begin{aligned}
&F_j^{(\mathrm{c})}(u,\partial_x u)\\
&=F_j^{(\mathrm{c})}(u,\partial_x u)-F_j^{(\mathrm{c})}\Big(u,\partial_x u+\frac{i}{t}Ju\Big)\\
&=\frac{-i}{t}\sum_{k=1}^{m}\Big(\int_0^1\frac{\partial F_j^{(\mathrm{c})}}{\partial q_k}\Big(u,\partial_x u+\frac{i\theta}{t}Ju\Big)\,d\theta\Big)Ju_k\\
&\quad+\frac{i}{t}\sum_{k=1}^{m}\Big(\int_0^1\frac{\partial F_j^{(\mathrm{c})}}{\partial\overline{q_k}}\Big(u,\partial_x u+\frac{i\theta}{t}Ju\Big)\,d\theta\Big)\overline{Ju_k},
\end{aligned}\tag{4}$$

which implies a gain of extra time-decay in the worst contribution of the nonlinear terms. Using this gain, we can derive an a priori estimate for

$$\begin{aligned}
E_s(T)=\sup_{t\in[0,T)}\Big\{&\sum_{j+k\le s-1}\|J^j\partial_x^k u(t,\cdot)\|_{L^2}\\
&+(1+t)^{-\mu}\sum_{j+k=s}\|J^j\partial_x^k u(t,\cdot)\|_{L^2}\Big\}
\end{aligned}$$

with $\mu\in(0,1/2)$ and $s\ge 5$, provided that $E_s(0)$ is small enough. More precisely, we have the following

Lemma 1. *Let u be a solution to (1)–(2) for $0\le t<T$. There are constants $\gamma_0\in(0,1]$, $\varepsilon>0$ and $C>0$, which are independent of T, such that $E_s(T)\le\gamma$ implies $E_s(T)\le C(\varepsilon+\gamma^{3/2})$, provided $\gamma\le\gamma_0$ and $\sum_{j+k\le s}\|x^j\partial_x^k\varphi\|_{L^2}\le\varepsilon$.*

Once this lemma is established, we can obtain the desired conclusion along the standard way. (For the local existence of NLS systems, see e.g. [KPV1, KPV2].)

Remark 2. The above argument is based on the idea of Katayama–Tsutsumi [KT], who considered the single NLS with the nonlinearity $(\lambda u+\mu\partial_x u)\partial_x|u|^2$. However, we would insist on the following two points: The first one is that their argument is not directly applicable for the systems because they relies on the gauge transformation technique to estimate the highest derivatives of u. Indeed, some basic identities are missed in the present situation, for instance

$$\partial_x e^{a(x)} \neq e^{a(x)} \partial_x a(x),$$

$$2\overline{v} \cdot b(x)\overline{\partial_x v} \neq \partial_x \big(b(x)\overline{v} \cdot \overline{v}\big) - \big(\partial_x b(x)\big)\overline{v} \cdot \overline{v}$$

if $a(x)$, $b(x)$ are matrix-valued and v is vector-valued in general. The second point (much more important) is that the identity $\partial_x |u|^2 = (\overline{u}Ju - u\overline{Ju})/(it)$, which first appeared in [T] and was effectively used in [KT], has never been interpreted in such a way that we do here.

Remark 3. From the viewpoint of global existence, our null condition is not optimal because it does not cover some trivial cases, such as

$$\begin{cases} \left(i\partial_t + \dfrac{1}{2}\partial_x^2\right) u_1 = F_1(u_2, \partial_x u_2), \\ \left(i\partial_t + \dfrac{1}{2}\partial_x^2\right) u_2 = 0, \end{cases} \tag{5}$$

which is practically an inhomogeneous linear equation. Nevertheless, our condition is fairly sharp as a condition for the solution being asymptotically free. In fact, if $F_1^{(c)}(v_0, i\xi_0 v_0) \neq 0$ for some $(\xi_0, v_0) \in \mathbf{R} \times \mathbf{C}$ in (5), then, taking the initial data φ_2 such that $\hat{\varphi}_2(\xi_0) = v_0$, we can prove that there exist constants $C_1 \geq 0$, $C_2 > 0$ and $T \geq 1$ such that

$$\|A\|_{L^2} \log t + C_1 \geq \|u_1(t, \cdot)\|_{L^2} \geq C_2 |A(\xi_0)| \log t$$

for all $t \geq T$, where $A(\xi) = -iF_1^{(c)}(\hat{\varphi}_2(\xi), i\xi\hat{\varphi}_2(\xi))$ and $\hat{\varphi}(\xi)$ denotes the Fourier transform of $\varphi(x)$. Since $|A(\xi_0)| = |F_1^{(c)}(v_0, i\xi_0 v_0)| > 0$, the above inequality tells us that the L^2 norm of u_1 grows like $\log t$ in the large value of t. In particular, we see that u_1 does *not* behave like a free solution in the sense of Theorem 1 because otherwise $\|u_1(t, \cdot)\|_{L^2}$ must stay bounded for all time.

3. Proof of Lemma 1. This section is devoted to the proof of Lemma 1. In what follows we denote various positive constants by C, which may be different line by line.

First we give some preliminaries which are needed in the proof. As is well-known, the usual energy inequality causes loss of derivatives when the nonlinear term contains derivatives of the unknowns. To overcome this obstacle, we shall use a smoothing property of the Schrödinger operator following [HNP] (see also [Chi1, Chi2, D], etc.). Let $\mathcal{H}$ be the Hilbert transform, that is,

$$\mathcal{H}v(x) = \frac{1}{\pi} \text{ p.v.}\int_{\mathbf{R}} \frac{v(y)}{x-y}\, dy$$

for $v \in L^2$. Also let us define the operator S_Φ by

$$\begin{aligned} S_\Phi v = &\left\{\cosh\Big(\int_{-\infty}^{x} \Phi(y)\, dy\Big)\right\} v \\ &+ i\left\{\sinh\Big(\int_{-\infty}^{x} \Phi(y)\, dy\Big)\right\} \mathcal{H}v \end{aligned}$$

with a non-negative function Φ. Note that S_Φ is L^2-automorphism and that both of $\|S_\Phi\|_{L^2 \to L^2}$, $\|S_\Phi^{-1}\|_{L^2 \to L^2}$ are dominated by $2\exp(\|\Phi\|_{L^1})$. We also remark that S_Φ satisfies, roughly, $[\partial_x^2, S_\Phi] \simeq 2i\Phi(x)S_\Phi|\partial_x|$ up to harmless remainders. This enables us to obtain the smoothing estimate of order $1/2$. More precisely, we have the following

Lemma 2 (Lemma 2.2 of [HNP]). *Let $v(t,x)$, $\Phi(t,x)$ be smooth functions with suitable decay as $|x| \to \infty$, and put $S = S_{\Phi(t,\cdot)}$, $g = i\partial_t v + \frac{1}{2}\partial_x^2 v$ We assume that Φ is non-negative and $\partial_x\sqrt{\Phi(t,\cdot)}$ belongs to L^∞. Then we have*

$$\begin{aligned} &\frac{d}{dt}\|Sv(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\int_{\mathbf{R}} \Phi(t,x)\Big|S|\partial_x|^{1/2} v(t,x)\Big|^2 dx \\ &\leq 2\,\mathrm{Im}\,\big\langle Sv(t,\cdot), Sg(t,\cdot)\big\rangle_{L^2} + CB(t)\|v(t,\cdot)\|_{L^2}^2, \end{aligned}$$

where

$$\begin{aligned} B(t) =& e^{2\|\Phi(t,\cdot)\|_{L^1}}\Big\{\|\Phi(t,\cdot)\|_{L^\infty} + \|\partial_x\sqrt{\Phi(t,\cdot)}\|_{L^\infty}^2 \\ &+ \|\Phi(t,\cdot)\|_{L^\infty}^3 + \sup_{x\in\mathbf{R}}\Big|\partial_t\int_{-\infty}^{x}\Phi(t,y)\,dy\Big|\Big\}, \end{aligned} \tag{6}$$

and the operator $|\partial_x|^{1/2}$ is interpreted as the Fourier multiplier:

$$|\partial_x|^{1/2} v(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{ix\xi}|\xi|^{1/2}\hat{v}(\xi)d\xi.$$

Using this smoothing estimate combined with the next two auxiliary lemmas, we can get rid of the derivative loss (as well as the loss of time-decay) coming from the nonlinear terms.

Lemma 3. *Let $w(t,x)$ be a smooth function with suitable decay as $|x| \to \infty$. Suppose that $B(t)$ is given by (6) with $\Phi = (|w|^2 + |\partial_x w|^2)/\delta$, $\delta > 0$. Then we have*

$$B(t) \leq e^{\frac{C}{\delta}\|w(t,\cdot)\|_{H^1}^2} \times$$

$$\times \frac{C}{\delta}\Big\{\|w(t,\cdot)\|_{W^{2,\infty}}^2+\delta^{-2}\|w(t,\cdot)\|_{W^{1,\infty}}^6 + \|w(t,\cdot)\|_{H^1}\Big\|\Big(i\partial_t+\frac{1}{2}\partial_x^2\Big)w(t,\cdot)\Big\|_{H^1}\Big\}.$$

Here the constant C is independent of w and δ.

Lemma 4. *Let $v = (v_1,\cdots,v_m)$, $w = (w_1,\cdots,w_m)$ be $\mathbf{C}^m$ valued smooth functions of x with suitable decay as $|x|\to\infty$. Suppose that for each $j,k\in\{1,\cdots,m\}$, $P_{jk}[w]$ is a function of $(w,\partial_x w)$ vanishing with quadratic order at $(0,0)$. Put $S=S_\Phi$ with $\Phi(x)=(|w(x)|^2+|\partial_x w(x)|^2)/\delta$, where δ is a positive constant. Then we have*

$$\begin{aligned}&\sum_{j,k=1}^m \big|\big\langle Sv_j, S(P_{jk}[w]\partial_x v_k)\big\rangle_{L^2}\big|\\ &\le C\delta\int_{\mathbf{R}}\Phi(x)\Big|S|\partial_x|^{1/2}v(x)\Big|^2dx\\ &+Ce^{\frac{C}{\delta}\|w\|_{H^1}^2}(1+\delta^{-2}\|w\|_{H^2}^4)\|w\|_{W^{2,\infty}}^2\|v\|_{L^2}^2\end{aligned}$$

and

$$\begin{aligned}&\sum_{j,k=1}^m \big|\big\langle Sv_j, S(P_{jk}[w]\overline{\partial_x v_k})\big\rangle_{L^2}\big|\\ &\le C\delta e^{\frac{C}{\delta}\|w\|_{H^1}^2}\int_{\mathbf{R}}\Phi(x)\Big|S|\partial_x|^{1/2}v(x)\Big|^2dx\\ &+Ce^{\frac{C}{\delta}\|w\|_{H^1}^2}(1+\delta^{-2}\|w\|_{H^2}^4)\|w\|_{W^{2,\infty}}^2\|v\|_{L^2}^2.\end{aligned}$$

Here the constant C does not depend on v, w and δ.

These lemmas are just the applications of the results of [HNP] to our situation.

Now, we are ready to prove Lemma 1. In what follows we write $Z^\alpha = J^{\alpha_1}\partial_x^{\alpha_2}$ for a multi-index $\alpha=(\alpha_1,\alpha_2)\in(\mathbf{N}\cup\{0\})^2$, and use the following notations:

$$|w(t,x)|_\sigma=\sum_{|\alpha|\le\sigma}|Z^\alpha w(t,x)|,$$
$$\|w(t)\|_{\sigma,2}=\big\||w(t,\cdot)|_\sigma\big\|_{L^2}$$

and

$$\|w(t)\|_{\sigma,\infty}=\big\||w(t,\cdot)|_\sigma\big\|_{L^\infty}$$

for $\sigma\ge 0$ and smooth function $w(t,x)$ decaying fast as $|x|\to\infty$. Remark that (3) implies

$$\|w(t)\|_{\sigma,\infty}\le\frac{C}{(1+t)^{1/2}}\|w(t)\|_{\sigma+1,2}.$$

Also we note that $Z^\alpha u_k$ satisfies

$$\Big(i\partial_t+\frac{1}{2}\partial_x^2\Big)Z^\alpha u_j = Z^\alpha F_j(u,\partial_x u)$$

for any α, if u solves (1), because of the commutation relation $\big[i\partial_t+\frac{1}{2}\partial_x^2, Z^\alpha\big]=0$.

First we consider the case where $|\alpha|\le s-1$. It follows from (4) that

$$\begin{aligned}&\|Z^\alpha F_j(u,\partial_x u)\|_{L^2}\\ &\le\|Z^\alpha F_j^{(\mathrm{c})}\|_{L^2}+\|Z^\alpha(F_j-F_j^{(\mathrm{c})})\|_{L^2}\\ &\le\frac{C}{1+t}\|u\|^2_{[\frac{|\alpha|+1}{2}]+1,\infty}\|u\|_{|\alpha|+1,2}\\ &\quad+\|u\|^3_{[\frac{|\alpha|+1}{2}]+1,\infty}\|u\|_{|\alpha|+1,2}\\ &\le\frac{C}{(1+t)^2}\|u\|^2_{s-1,2}\|u\|_{s,2}\\ &\quad+\frac{C}{(1+t)^{3/2}}\|u\|^3_{s-1,2}\|u\|_{s,2}\\ &\le\frac{C\gamma^3}{(1+t)^{2-\mu}}+\frac{C\gamma^4}{(1+t)^{3/2-\mu}}.\end{aligned}$$

So the standard energy inequality implies

$$\begin{aligned}&\sup_{0\le t<T}\sum_{|\alpha|\le s-1}\|Z^\alpha u(t,\cdot)\|_{L^2}\\ &\le C\varepsilon+C\gamma^3\int_0^T\frac{d\tau}{(1+\tau)^{3/2-\mu}}\\ &\le C(\varepsilon+\gamma^3).\end{aligned}\tag{7}$$

Next we consider the case where $|\alpha|=s$. We shall apply Lemma 2 with $v=Z^\alpha u$, $w=u$, $\delta=\gamma$ to obtain

$$\begin{aligned}&\frac{d}{dt}\|SZ^\alpha u_j\|_{L^2}^2+\frac{1}{2}\int_{\mathbf{R}}\Phi\Big|S|\partial_x|^{1/2}Z^\alpha u_j\Big|^2dx\\ &\le 2\,\mathrm{Im}\,\big\langle SZ^\alpha u_j, SZ^\alpha F_j(u,\partial_x u)\big\rangle_{L^2}\\ &\quad+CB(t)\|Z^\alpha u_j\|_{L^2}^2.\end{aligned}$$

To estimate the first term of the right hand side, we set

$$P_{jk}[u]=\frac{\partial F_j}{\partial q_k}(u,\partial_x u),\quad Q_{jk}[u]=\frac{\partial F_j}{\partial\overline{q_k}}(u,\partial_x u)$$

so that $Z^\alpha F_j(u,\partial_x u)$ splits into

$$\begin{aligned}Z^\alpha F_j=&\sum_{k=1}^m P_{jk}[u]\partial_x(Z^\alpha u_k)\\ &+(-1)^{\alpha_1}\sum_{k=1}^m Q_{jk}[u]\overline{\partial_x(Z^\alpha u_k)}+R\end{aligned}$$

with

$$\|R\|_{L^2} \le \frac{C}{1+t}\|u\|_{s-1,2}^2\|u\|_{s,2}$$

(cf. Lemma 2.2 of [KT]). Note that $[\frac{s}{2}]+1 \le s-2$, for $s \ge 5$. Then we have

$$\begin{aligned}
&\sum_{j=1}^{m} \operatorname{Im}\left\langle SZ^\alpha u_j, SZ^\alpha F_j(u,\partial_x u)\right\rangle_{L^2} \\
&\le \sum_{j,k=1}^{m} \left|\left\langle SZ^\alpha u_j, S(P_{jk}[u]\partial_x Z^\alpha u_k)\right\rangle_{L^2}\right| \\
&\quad + \sum_{j,k=1}^{m} \left|\left\langle SZ^\alpha u_j, S(Q_{j,k}[u]\overline{\partial_x Z^\alpha u_k})\right\rangle_{L^2}\right| \\
&\quad + \sum_{j=1}^{m} \|SZ^\alpha u_j\|_{L^2}\|SR\|_{L^2} \\
&\le C\gamma e^{\frac{C}{\gamma}\|u\|_{1,2}^2} \int_{\mathbf{R}} \Phi \left| S|\partial_x|^{1/2} Z^\alpha u\right|^2 dx \\
&\quad + Ce^{\frac{C}{\gamma}\|u\|_{1,2}^2}(1+\gamma^{-2}\|u\|_{2,2}^4)\|u\|_{2,\infty}^2\|u\|_{s,2}^2 \\
&\quad + \frac{Ce^{\frac{C}{\gamma}\|u\|_{1,2}^2}}{1+t}\|u\|_{s-1,2}^2\|u\|_{s,2}^2 \\
&\le C\gamma \int_{\mathbf{R}} \Phi \left| S|\partial_x|^{1/2} Z^\alpha u\right|^2 dx + \frac{C\gamma^4}{(1+t)^{1-2\mu}}.
\end{aligned}$$

Here we have used Lemma 4. On the other hand, we apply Lemma 3 to obtain

$$\begin{aligned}
B(t) \le& \frac{Ce^{\frac{C}{\gamma}\|u\|_{1,2}^2}}{\gamma}\left\{\|u\|_{2,\infty}^2\right. \\
&\left. + \gamma^{-2}\|u\|_{1,\infty}^6 + \|u\|_{1,2}\|F(u,\partial_x u)\|_{H^1}\right\} \\
\le& \frac{C\gamma}{1+t},
\end{aligned}$$

whence

$$B(t)\|Z^\alpha u_j\|_{L^2}^2 \le \frac{C\gamma^3}{(1+t)^{1-2\mu}}.$$

Summing up, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|SZ^\alpha u\|_{L^2}^2 \\
&\le \left(C\gamma - \frac{1}{2}\right)\int_{\mathbf{R}} \Phi\left|S|\partial_x|^{1/2} Z^\alpha u\right|^2 dx + \frac{C\gamma^3}{(1+t)^{1-2\mu}} \\
&\le \frac{C\gamma^3}{(1+t)^{1-2\mu}},
\end{aligned}$$

provided γ is so small that $C\gamma - \frac{1}{2} \le 0$. Integrating with respect to t, we have

$$\begin{aligned}
\|SZ^\alpha u(t,\cdot)\|_{L^2}^2 &\le C\varepsilon^2 + C\gamma^3 \int_0^t \frac{d\tau}{(1+\tau)^{1-2\mu}} \\
&\le C\varepsilon^2 + C\gamma^3(1+t)^{2\mu},
\end{aligned}$$

which yields

$$\begin{aligned}
&\sup_{0\le t<T}(1+t)^{-\mu}\sum_{|\alpha|=s}\|Z^\alpha u(t,\cdot)\|_{L^2} \\
&\le \sup_{0\le t<T}(1+t)^{-\mu} e^{\frac{C}{\gamma}\|u\|_{1,2}^2}\sum_{|\alpha|=s}\|SZ^\alpha u\|_{L^2} \qquad (8)\\
&\le C(\varepsilon + \gamma^{3/2}).
\end{aligned}$$

Combining (7) and (8), we obtain

$$E_s(T) \le C(\varepsilon + \gamma^{3/2}). \qquad \square$$

4. Concluding Remark. As we have mentioned in Remark 3, our null condition is not optimal for small data global existence of (1)–(2). In view of the recent progress on NLW ([A2, A3, L, LR1, LR2] etc.), it would be natural to introduce a weaker condition, which is to be called the weak null condition. Recently, such an attempt is made in [Su] and [HNS] for the single NLS (i.e., the case of $m = 1$). According to them, an NLS analogue of the weak null condition could be given by

$$\sup_{\xi\in\mathbf{R}} \operatorname{Im} F_1^{(c)}(1, i\xi) \le 0 \qquad (9)$$

in the single case (see [Su] and [HNS] for the details). When $m \ge 2$, some variant of their argument suggests that large time behavior of the solution to (1)–(2) might be characterized by the behavior (as $\tau \to +\infty$) of the solution $v(\tau) = (v_k(\tau;\xi))_{1\le k\le m}$ to the ODE

$$\frac{dv_k}{d\tau} = -iF_k^{(c)}(v, i\xi v) \quad (k = 1, \cdots, m), \qquad (10)$$

where $\xi \in \mathbf{R}$ is regarded as a parameter. Note that (10) is reduced to

$$\frac{dv_1}{d\tau} = -iF_1^{(c)}(1, i\xi)|v_1|^2 v_1$$

when $m = 1$. This gives a heuristic reason why (9) appears. However, we are presently not able to justify this expectation for general NLS systems.

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