Harmonic Teichmüller mappings

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Abstract: In this paper we present a necessary and sufficient condition for a $C^2$ Teichmüller mapping to be $\rho$-harmonic. By this result we show that there is no solution to the Schoen conjecture in the class of $C^2$ Teichmüller mappings. We also obtain two characterizations for $\pi$-harmonic Teichmüller mappings.

Key words: Quasiconformal mappings; Teichmüller mappings; harmonic mappings.

1. Introduction. Let $\Omega$ and $\Omega'$ be two Jordan domains of the complex plane $\mathbb{C}$. $\rho(w)|dw|^2$ is a singular metric of $\Omega'$ if its metric density $\rho$ is a non-negative smooth function and admits isolated zeros on $\Omega'$. Given a holomorphic quadratic differential $\psi(w)dw^2$ on $\Omega'$, $|\psi(w)||dw|^2$ defines a singular flat metric with singularities at the zeros of $\psi$. In this paper, a $C^2$ mapping $f : \Omega \to \Omega'$ is said to be harmonic with respect to $\rho$ (shortly $\rho$-harmonic) if it satisfies

$$\partial z (\rho(\phi) f_z \bar{f}_z) = 0,$$

that is, the Hopf differential $\rho(\phi) f_z \bar{f}_z dz^2$ of $f$ is holomorphic in $\Omega$. Gromov and Schoen [4] first systematically studied harmonic mappings with respect to a singular metric in connection with arithmetic superrigidity.

If a $\rho$-harmonic mapping $f$ from $\Omega$ onto $\Omega'$ is also quasiconformal, then we call it a $\rho$-harmonic quasiconformal mapping (abbreviated $\rho$-HQC mapping). Particularly, we say that $f$ is a $\pi$-HQC mapping if $\rho$ is a constant. It is well known that the Jacobian $J_f$ of each $\rho$-HQC mapping $f$ is always positive since $f$ is a $C^2$ quasiconformal mapping. The Beltrami coefficient $\mu_f$ of a $\rho$-HQC mapping $f$ is of the form

$$\mu_f = |\mu_f| \frac{\varphi}{|\varphi|},$$

where $\varphi = \rho(\phi) f_z \bar{f}_z$ (see [16]).

A $\rho$-HQC mapping $f$ is a Teichmüller mapping if $|\mu_f|$ is a constant $k$ with $0 < k < 1$, then we call $f$ a $\rho$-harmonic Teichmüller mapping and $\varphi(z)dz^2$ the associated holomorphic quadratic differential of $f$. In particular, $f$ is said to be a $\pi$-harmonic Teichmüller mapping if $\rho$ is a constant. Let $\phi(w)|dw|^2$ be the associated holomorphic quadratic differential of $f^{-1}$, then $f$ induces a singular flat metric $|\phi(w)||dw|^2$ on $\Omega'$. One can refer to [5, 6] for some results about harmonic Teichmüller mappings with respect to singular flat metrics.

Beurling and Ahlfors [2] proved that a homeomorphism from the unit circle onto itself admits a quasiconformal extension from the unit disk onto itself if and only if it is quasisymmetric. In order to prove that there always exists an extension with good properties, Schoen [19] posed the following conjecture.

Schoen conjecture. The harmonic quasiconformal homeomorphisms with respect to the Poincaré metric from the unit disk onto itself are parameterized by the set of quasisymmetric homeomorphisms of the unit circle onto itself.

There are some partial results on the above conjecture. When the boundary mapping is also sufficiently differentiable, Li and Tam [7] proved that there exists a HQC mapping with respect to the Poincaré metric. Later, Li [8] and Markovic [11] generalized their result. Since Teichmüller mappings play a vital role in the theory of quasiconformal mappings, the following question is natural.

Question 1. Does there exist a solution to the Schoen conjecture in the class of $C^2$ Teichmüller mappings?

It is equivalent to ask whether a Teichmüller mapping can be a HQC mapping with respect to the Poincaré metric. Namely, does there exist a HQC
mapping with respect to the Poincaré metric such that the modulus of its Beltrami coefficient is a constant? In order to deal with it, we only need to answer the following question.

**Question 2.** For a $C^2$ Teichmüller mapping $f$, does there uniquely exist a metric $\rho(w)|dw|^2$ such that $f$ is $\rho$-harmonic?

In Section 2, we will answer Question 2 (see Theorem 2.1). In fact we will prove that a $C^2$ Teichmüller mapping $f$ is harmonic with respect to a singular flat metric, which is determined by the associated holomorphic quadratic differential of $f^{-1}$. Then we show that such metric is unique ignoring multiplying a positive constant. As an application of Theorem 2.1 to the Schoen conjecture, we will give a negative answer to Question 1 in Section 3.

Martio [12] first studied the existence of $\pi$-HQC mappings. Recently, there is much progress in prescribing the properties of $\pi$-HQC mappings in [3, 9, 10, 13–15]. As another application of Theorem 2.1, we will prescribe some properties of a $\pi$-harmonic Teichmüller mapping in Section 4. It is well known that a univalent $\pi$-harmonic mapping $f$ has a canonical representation $f = h + \bar{g}$ where $g$ and $h$ are holomorphic. Furthermore, $g = \bar{a}h$ if $f$ is also a Teichmüller mapping, where $h$ is a conformal mapping and $\alpha$ is a constant such that $0 < |\alpha| < 1$.

2. A necessary and sufficient condition for a $C^2$ Teichmüller mapping to be $\rho$-harmonic. In this section, we will prove the existence for singular metrics by Lemma 2.1, and then obtain a sufficient condition for a $C^2$ Teichmüller mapping to be $\rho$-harmonic. By Lemma 2.2, we show that the condition is also necessary.

**Lemma A.** ([17, 18]) If $f$ is a Teichmüller mapping from the unit disk $\Delta$ onto itself then its inverse mapping $g = f^{-1}$ is also a Teichmüller mapping.

**Remark.** For a Teichmüller mapping from $\Omega$ onto $\Omega'$ the result of Lemma A also holds.

**Lemma 2.1.** If $f$ is a $C^2$ Teichmüller mapping from $\Omega$ onto $\Omega'$ and the associated holomorphic quadratic differential of its inverse mapping $\phi(w)dw^2$ then $f$ is $c|\phi|$-harmonic, where $c$ is a positive constant.

**Proof.** Denote by $F$ the inverse mapping of $f$. We know that $F$ is also a Teichmüller mapping by Lemma A. Let $\phi(w)dw^2$ be the associated holomorphic quadratic differential of $F$. Set $\psi^2(w) = \phi(w).$ Then there exists a constant $k$, $0 < k < 1$, such that

$$\mu_F = \frac{F_{\bar{w}}}{F_w} = k \frac{\psi}{|\phi|} = k \frac{\psi^2}{|\psi|^2} = k \frac{\psi}{\psi^2}.$$  

By differentiating both sides of (2.1) in $w$, we have

$$F_{ww} = \frac{F_{w\bar{w}}\psi}{k\psi} + \frac{F_{w\bar{w}}}{k\psi}.$$  

Hence

$$(c|\phi(f)||f_z|^2)z = -\frac{ck^2}{2(1-k^2)^2} \frac{|\phi|^2}{F_wF_{\bar{w}}} f(z).$$

Since $F_w \circ f = f_z/J_f$ and $-F_{\bar{w}} \circ f = f_{\bar{z}}/J_f$ (see [1, p. 8]), we have

$$(c|\phi(f)||f_z|^2)z = \frac{2ck}{(1-k^2)^2} \frac{\psi^2F_w^2F_{\bar{w}} - \psi^2F_wF_{\bar{w}}F_{ww} + \psi^2F_w^2F_{\bar{w}}}{J_F F_{\bar{w}}} f(z).$$

Therefore $f$ is a $c|\phi|$-harmonic mapping, and the proof is completed.

Next we will prove that the metric is unique ignoring multiplying a positive constant.

**Lemma 2.2.** If $f$ is a $\rho$-harmonic Teichmüller mapping from $\Omega$ onto $\Omega'$, then

$$\rho(w) = c|\phi(w)| \quad (w \in \Omega'),$$

where $\phi(w)dw^2$ is the associated holomorphic quadratic differential of $f^{-1}$, and $c$ is a positive constant.

**Proof.** By the hypothesis we have

$$(\rho(f)F_zF_{\bar{z}})z = 0 \quad (z \in \Omega).$$

Let $F$ be the inverse mapping of $f$. We assume that $J_f$ and $J_F$ are the Jacobians of $f$ and $F$, respectively. Since $F_w \circ f = f_z/J_f$ and $-F_{\bar{w}} \circ f = f_{\bar{z}}/J_f$, the relation (2.5) is equivalent to

$$(\psi^2F_wF_{\bar{w}}/J_F \circ f)z = 0 \quad (w = f(z), \quad z \in \Omega).$$
By the chain rule we have

\begin{equation}
(\rho F_w, F_w ϕ) = (-\frac{F_w}{F_φ}) + \rho \frac{F_w}{F_φ} (F_w, F_w) = 0 \quad (w \in \Omega').
\end{equation}

Lemma A implies that \( F \) is also a Teichmüller mapping from \( Ω' \) onto \( Ω \). Set \( φ(w) = ψ^2(w) \), then the Beltrami coefficient of \( F \) is of the form

\begin{equation}
μ_F = \frac{F_w}{F_φ} = \frac{k}{|φ|} = \frac{k|ψ|^2}{|ψ|} = \frac{k}{ψ}.
\end{equation}

By the relation \( \frac{k^2}{μ_F} = μ_F \), we reduce (2.7) to

\begin{equation}
μ_F(\frac{ρ}{F_w F_φ}) = (\frac{F_w}{F_φ}) ρ.
\end{equation}

Hence

\begin{equation}
ρ F_w F_φ - ρ (F_φ F_φ F_w + F_w F_φ) = \frac{ρ}{F_φ} = \frac{ρ}{F_w F_φ}.
\end{equation}

By direct calculations it follows that

\begin{equation}
(μ_F ρ_w - ρ_φ) F_w F_φ + ρ (F_w F_φ F_w + F_w F_φ) = 0.
\end{equation}

From (2.8) we have

\begin{equation}
F_w = \frac{F_φ}{k ψ} + \frac{F_w ψ'}{k ψ}.
\end{equation}

and

\begin{equation}
F_w = \frac{k F_φ ψ'}{ψ} + \frac{k F_w ψ'}{ψ}.
\end{equation}

Thus by (2.10) and (2.11), the relation (2.9) can be simplified as

\begin{equation}
μ_F ρ_w - ρ_φ = μ_F ρ_w = \frac{μ_F ψ'}{ψ} - \frac{ρ_φ}{ψ}.
\end{equation}

Since \( ρ \) is a real function we obtain

\begin{equation}
μ_F(\log \frac{ρ}{ψ}) = (\log \frac{ρ}{ψ}) ψ.
\end{equation}

Noting that \( 0 < |μ_F| = k < 1 \), we have

\begin{equation}
(\log \frac{ρ}{ψ}) ψ = 0.
\end{equation}

that is,

\begin{equation}
(\log \frac{ρ}{ψ}) ψ = 0.
\end{equation}

So \( ρ/ψ \) is analytic, namely, \( \frac{ρ}{|ψ|^2} ψ \) is analytic. Therefore \( ρ/|ψ|^2 \) (i.e., \( ρ/|φ| \)) is a positive constant.

Combining Lemma 2.1 with Lemma 2.2, we obtain

**Theorem 2.1.** If \( f \) is a \( C^2 \) Teichmüller mapping from \( Ω \) onto \( Ω' \) and the associated holomorphic quadratic differential of its inverse mapping is \( φ(w) dw^2 \), then \( f \) is a \( ρ \)-harmonic mapping if and only if \( ρ = c|φ| \), where \( c \) is a positive constant.

3. \( ρp \)-harmonic Teichmüller mappings.

In this section we will give a negative answer to Question 1. In fact we will obtain a more general result (see Theorem 3.1). The Poincaré metric \( ρp |dz|^2 \) of the unit disk of \( Δ \) is \( |dz|^2/(1-|z|^2)^2 \). Generally, the Poincaré metric \( ρp |dw|^2 \) of \( Ω \) is given by \( |φ'(w)|^2|dw|^2/(1-|φ(w)|^2)^2 \), where \( φ \) is a conformal mapping from \( Ω \) onto \( Δ \). Let \( Ω_i \) \((i = 0, 1, 2, 3) \) be Jordan domains of the complex plane \( C \).

**Lemma 3.1.** Let \( φ \) be a conformal mapping from \( Ω_2 \) onto \( Ω_3 \) and \( φ \) be a conformal mapping from \( Ω_0 \) onto \( Ω_1 \). If \( F : Ω_1 \to Ω_2 \) is harmonic with respect to \( ρ_2 \), then \( φ \circ F \) is harmonic with respect to \( ρ_3 \), and \( F \circ φ \) is harmonic with respect to \( ρ_2 \), where \( ρ_1 \) is the Poincaré metric density of \( Ω_i \) \((i = 2, 3) \), respectively.

**Proof.** If \( F \) is harmonic with respect to \( ρ_2 \), then by definition \( ρ_2(F(z)) F_z F_φ \) is holomorphic in \( Ω_1 \). If \( φ \) is a conformal mapping from \( Ω_2 \) onto \( Ω_3 \), then \( φ \circ F \) is a \( C^2 \) homeomorphism from \( Ω_1 \) onto \( Ω_3 \) and

\begin{equation}
ρ_3(φ \circ F)(φ \circ F) \frac{φ'}{φ} = ρ_3(φ \circ F)(φ' \circ F)^2 F_z F_φ \frac{φ'}{φ} = ρ_3(φ \circ F)(φ' \circ F)^2 F_z F_φ = ρ_2(F(z)) F_z F_φ,
\end{equation}

where \( w = F(z) \). Thus, \( ρ_3(φ \circ F)(φ \circ F) \frac{φ'}{φ} \) is holomorphic in \( Ω_1 \), that is, \( φ \circ F \) is harmonic with respect to \( ρ_3 \).

Let \( φ \) be a conformal mapping from \( Ω_0 \) onto \( Ω_1 \). We therefore obtain that \( F \circ φ \) is a \( C^2 \) homeomorphism of \( Ω_0 \) onto \( Ω_2 \) and

\begin{equation}
ρ_2(F \circ φ(ζ)) F_z(φ(ζ)) F_z(φ(ζ)) = ρ_2(F \circ φ(ζ)) F_z(φ(ζ)) F_z(φ(ζ)) = ρ_2(F(z)) F_z(z) F_z(z) = ρ_2(F(z)) F_z(z) F_z(z).
\end{equation}
where \( z = \varphi(\zeta) \). Since \( \rho_2(F)F_z \bar{F}_{\bar{z}} \) is holomorphic, we can write \( \psi(z) = \rho_2(F)F_z \bar{F}_{\bar{z}} \). So \( \psi(\varphi(\zeta)) \) is holomorphic in \( \Omega_0 \). Thus \( \psi(\varphi(\zeta))(\varphi'(\zeta))^2 \) is holomorphic in \( \Omega_0 \), that is, \( \rho_2(F \circ \varphi(\zeta))(F \circ \varphi)'(\varphi \circ \varphi)_{\zeta} \) is holomorphic in \( \Omega_0 \). Hence \( F \circ \varphi \) is harmonic with respect to \( \rho_2 \).

**Theorem 3.1.** There does not exist any \( \rho_p \)-harmonic Teichmüller mapping from \( \Omega \) onto \( \Omega' \), where \( \rho_p \) is the Poincaré metric density of \( \Omega' \).

**Proof.** By Lemma 3.1, we can choose \( \Omega' \) to be the unit disk \( \Delta \). Suppose that there exists a \( \rho_p \)-harmonic Teichmüller mapping \( \varphi \) of \( \Omega \) onto \( \Delta \). Then from Theorem 2.1, there exists a holomorphic function \( \phi(w) \), \( w \in \Delta \) such that

\[
(3.1) \quad 1/(1 - |w|^2)^2 = |\phi(w)|,
\]

where \( c \) is a positive constant.

Write \( \psi^2(w) = \phi(w) \). Differentiating both sides of (3.1) in \( w \), we obtain

\[
(3.2) \quad \frac{c\psi'}{\psi} = \frac{2w}{\psi (1 - |w|^2)^3}.
\]

By differentiating in \( w \) again, we have

\[
\frac{2}{\psi (1 - |w|^2)^3} (2 + \frac{\psi'}{\psi} w - \frac{3}{1 - |w|^2} \psi') = 0.
\]

Thus

\[
(3.3) \quad \frac{\psi'}{\psi} w = \frac{3}{1 - |w|^2} - 2.
\]

The left side of (3.3) is a meromorphic function but the right side of (3.3) is a real function and is not a constant, a contradiction. The proof is completed.

**4. \( \pi \)-harmonic Teichmüller mappings.**

In this section we will give another application of Theorem 2.1 to present some characterizations of \( \pi \)-harmonic Teichmüller mappings.

**Theorem 4.1.** If \( f \) is a \( C^2 \) Teichmüller mapping from \( \Omega \) onto \( \Omega' \) then the following conditions are equivalent

(i) \( f \) is \( \pi \)-harmonic;

(ii) the Beltrami coefficient of its inverse mapping is a constant;

(iii) \( f \) is of the form \( f = h + \alpha \bar{h} \), where \( \alpha \) is a constant with \( 0 < |\alpha| < 1 \) and \( h \) is a conformal mapping in \( \Omega \).

**Proof.** It is obvious that (iii) implies (i). Suppose that \( f \) is a \( C^2 \) Teichmüller mapping from \( \Omega \) onto \( \Omega' \). Let \( F \) be the inverse of \( f \) and \( \phi(w)dw^2 \) the associated holomorphic quadratic differential of \( F \). It follows from Theorem 2.1 that \( f \) is \( \pi \)-harmonic if and only if \( \phi \) is a constant. Thus we see that (i) is equivalent to (ii).

Let \( \mu_F \) be the Beltrami coefficient of the inverse mapping \( F \). Suppose that \( \mu_F \) is a constant \(-b\) with \( |b| = k \). Using the formulas \( F_w \circ f = \frac{f_z}{f_{\bar{z}}}/J_f \) and \( -F_{\bar{w}} \circ f = \frac{f_{\bar{z}}}{f_z}/J_f \) again, we have

\[
\frac{f_{\bar{z}}}{f_z} = \mu_F \circ f = -b.
\]

Since \( f \) is a univalent \( \pi \)-harmonic mapping, there exist two holomorphic functions \( m \) and \( n \) such that \( f = m + \bar{n} \). Thus

\[
(4.1) \quad n' = bm'.
\]

So \( n = bm + d \), where \( d \) is a constant. Therefore

\[
f = m + bm + d. \quad \text{Set} \quad h = m + (d - bd)/(1 - |b|^2).
\]

Then there exists a holomorphic function \( h \) such that \( f = h + \bar{b}h \), where \( |b| = k \) and \( 0 < k < 1 \).

For two arbitrary points \( z_1 \) and \( z_2 \) in \( \Omega \), we get

\[
f(z_1) - f(z_2) = h(z_1) - h(z_2) + b(h(z_1) - h(z_2)).
\]

Hence \( f \) is univalent if and only if \( h \) is conformal in \( \Omega \). Therefore there exists a conformal mapping \( h \) in \( \Omega \) such that \( f = h + \bar{b}h \), where \( |b| = k \) and \( 0 < k < 1 \). Thus (ii) implies (iii).

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**References**


