A remark on continuous, nowhere differentiable functions

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Abstract: We consider a parameterized family of continuous functions, which contains as its members Bourbaki's and Perkins's nowhere differentiable functions as well as the Cantor-Lebesgue singular functions.

Key words: Continuous; nowhere differentiable function.

1. Introduction. Many examples are known of continuous, nowhere differentiable functions (see, for instance, [6, 10, 12]), notably Weierstrass's function and Takagi's function and their generalizations [4]. Of a different kind are a function by Bourbaki [1] and one by Perkins [9], which we would like to generalize in what follows. Although there have been written many papers on continuous, nowhere differentiable functions, our construction below seems to be one of the simplest and, at the same time, it discloses in a very elementary way a connection between nowhere differentiable functions and the Cantor-Lebesgue singular functions. In this respect, it seems to the author that the fact in the present paper may be worthy of notice.

We first fix a parameter $a \in (0,1)$. Then we define piecewise affine functions $\{f_n\}_{n=0}^{\infty}$ on the unit interval $[0,1]$ as follows. We start with $f_0(x) = x$. Suppose that f_n has been so defined that it is continuous on $[0,1]$, and is affine in each subinterval $k/3^n \le x \le (k+1)/3^n$, where $k = 0, 1, \ldots, 3^n - 1$. f_{n+1} is then defined by requiring: (1) f_{n+1} is continuous on $[0,1]$; (2) f_{n+1} is affine in each interval $k/3^{n+1} \le x \le (k+1)/3^{n+1}$, where $k = 0, 1, \ldots, 3^{n+1} - 1$; and (3) the following conditions hold true:

$$f_{n+1}\left(\frac{k}{3^n}\right) = f_n\left(\frac{k}{3^n}\right),$$

$$f_{n+1}\left(\frac{3k+1}{3^{n+1}}\right) = f_n\left(\frac{k}{3^n}\right) + a\left[f_n\left(\frac{k+1}{3^n}\right) - f_n\left(\frac{k}{3^n}\right)\right],$$

$$f_{n+1}\left(\frac{3k+2}{3^{n+1}}\right) = f_n\left(\frac{k}{3^n}\right) + (1-a)\left[f_n\left(\frac{k+1}{3^n}\right) - f_n\left(\frac{k}{3^n}\right)\right],$$

$$f_{n+1}\left(\frac{k+1}{3^n}\right) = f_n\left(\frac{k+1}{3^n}\right).$$

for $k = 0, 1, \ldots, 3^n - 1$. The operation from f_n to f_{n+1} is visualized in Fig. 1. We now define

$$F_a(x) = \lim_{n\to\infty} f_n(x).$$

As we will prove in the next section, F_a is continuous on $[0,1]$.

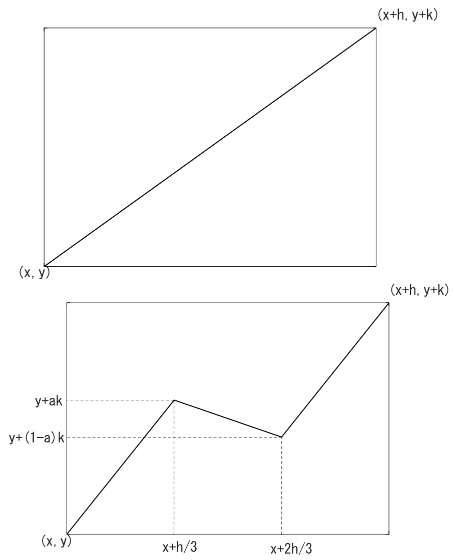

Fig. 1. The operation from f_n to f_{n+1}. Before the operation (top). After the operation (bottom). This operation is performed in each subinterval $[k/3^n, (k+1)/3^n]$.

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F_a becomes some known functions when a takes particular values. If $a = 5/6$, F_a is nothing but the function defined by Perkins [9]. This is a continuous, nowhere differentiable function (see Fig. 2, top). If $a = 2/3$, then it is the function defined by Bourbaki[1]. Both functions have some similarity with the function defined by Bolzano in 1830's (see [8]), but Bourbaki refers only to Bolzano and not to Perkins; Perkins refers only to the examples of Weierstrass, Faber, etc., but these examples are of nature different from $F_{5/6}$. Presumably Perkins came up with his function by himself with few hints from literatures.

If $a = 1/2$, then F_a becomes the Cantor-Lebesgue singular function, which is nondecreasing but has zero derivative almost everywhere. The value $a = 0$ can also be considered, in which case F_a becomes the Heaviside function:

$$F_0(x) = \begin{cases} 0 & (0 \le x < 1/2) \\ 1/2 & (x = 1/2) \\ 1 & (1/2 < x \le 1). \end{cases}$$

But this is discontinuous. Also, we see easily that F_1 is discontinuous, too. By these observations, we restrict ourselves to the case where $0 < a < 1$. Finally, we obviously have $F_{1/3}(x) \equiv x$.

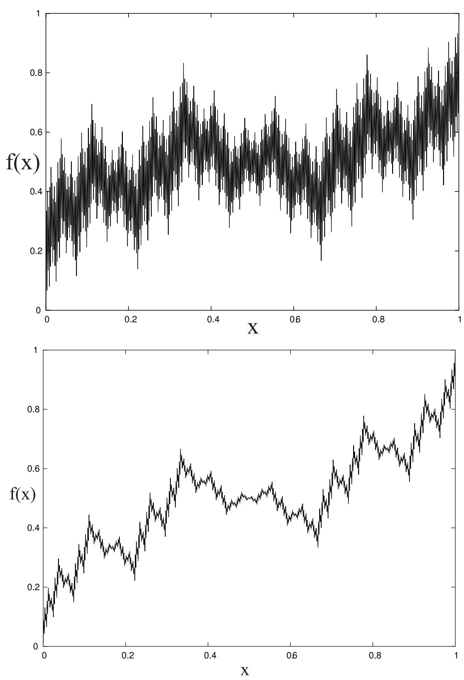

Fig. 2. The graph of Perkins's function (top) and that of Bourbaki's (bottom).

2. Properties of F_a.

Proposition 1. *For all* $0 < a < 1$, F_a *is well-defined and continuous everywhere on* $[0, 1]$.

Proof. Note first that $F_a(x) = f_n(x)$ if $x = k/3^n$ for $n = 1, 2, \ldots; k = 0, 1, \ldots, 3^n$. Let $A = \max\{a, |2a - 1|\}$. Clearly $0 < A < 1$. Note then that for any $n \ge 1$, the derivative of f_n satisfies

$$|f_n'(x)| \le (3A)^n$$

at all the points where f_n' exists. Note also that

$$\min\{f_n(k/3^n), f_n((k+1)/3^n)\} \\ \le f_{n+p}(x) \le \max\{f_n(k/3^n), f_n((k+1)/3^n)\}$$

for all $p = 1, 2, \ldots$ and $k/3^n \le x \le (k+1)/3^n$. Now, for any $x \in [0, 1]$ and any $\epsilon > 0$, take n and k such that $A^n < \epsilon$ and $k/3^n \le x < (k+1)/3^n$. Then

$$|f_{n+p}(x) - f_{n+q}(x)| \le A^n < \epsilon$$

for all $p, q \ge 0$. Since n is chosen independently of x, $\{f_n(x)\}$ converges uniformly. This proves our proposition. □

As for differentiability, the following is easy to prove:

Theorem 1. *If* $a \le 1/2$, *then the function* F_a *is nondecreasing. In particular, it is differentiable almost everywhere.*

As is often used effectively in the differentiability test, the following lemma will be used in what follows:

Lemma 1. *If* f *is differentiable at* x, *then*

$$\lim_{h\downarrow 0,\ k\downarrow 0} \frac{f(x+k) - f(x-h)}{k+h} = f'(x).$$

(*What is claimed is that the left hand side exists and is equal to the right hand side.*)

We now prove the following

Theorem 2. *If* $2/3 \le a < 1$, *then* F_a *is continuous on* $[0, 1]$, *but is nowhere differentiable.*

Proof. The proof by Perkins [9], where the case of $a = 5/6$ was considered, can be used with a minor change in the case where $2/3 < a < 1$. (The case of $a = 2/3$ will be considered later.) Note that $A = a$ in the present case. Then we easily see that

$$(3(2a-1))^n \le |f_n'(x)| \le (3a)^n$$

wherever $f_n'(x)$ exists. Since $2/3 < a$, $|f_n'|$ tends to infinity uniformly in x. Now let $x \in [0, 1]$ be arbitrarily chosen. For an arbitrary large n, we may choose an integer k such that $k/3^n \le x < (k+1)/3^n$. It then holds that

$$\left|\frac{F_a((k+1)/3^n) - F_a(k/3^n)}{1/3^n}\right| = \left|\frac{f_n((k+1)/3^n) - f_n(k/3^n)}{1/3^n}\right| \geq (3(2a-1))^n,$$

which tends to infinity. This shows, with the aid of Lemma 1, the nondifferentiability at x.

If $a = 2/3$, the above argument must be modified slightly. In this case, $|f_n'(x)|$ can remain bounded at some points, say for instance at $x = 1/2$. But even on such points where $f_n'(x)$ are bounded, we see that $|f_n'(x)| \geq 1$ and that $\{f_n'(x)\}$ changes sign infinitely often. Therefore,

$$\frac{f_n((k+1)/3^n) - f_n(k/3^n)}{1/3^n}$$

cannot converge to a definite value. □

Theorem 3. *If $1/2 < a < 2/3$, F_a is differentiable at infinite number of points. Also, it has no finite derivative at another set of infinite points.*

Proof. Let $x = k/3^n$, where n is a positive integer and $k = 0, 1, \ldots, 3^n$. We prove that F_a is nondifferentiable at such x's. We may assume without loss of generality that k is not a multiple of 3. If $k = 1$ modulo 3, then, by an elementary inspection as in the proof of the previous theorem, we have $D_- F_a(x) = +\infty$, where $D_- F_a$ denotes the left derivative. (Here $a > 1/3$ is enough.) If $k = 2$ modulo 3, then $D_+ F_a(x) = +\infty$, where $D_+ F_a$ denotes the right derivative. Also, $D_+ F_a(0) = +\infty, D_- F_a(1) = +\infty$.

Differentiability of F_a at other x's depends on the number theoretic property of x. Let x be a number not of the form $k/3^n$, and we consider the ternary expansion of x:

$$x = \frac{\xi_1}{3} + \frac{\xi_2}{3^2} + \frac{\xi_3}{3^3} + \cdots, \tag{1}$$

where $\xi_j = 0, 1$, or 2. Let $i(n)$ denote the number of those ξ_i $(i = 1, 2, \ldots, n)$ such that $\xi_i = 1$. Then $f_n'(x) = (3b)^{i(n)}(3a)^{n-i(n)}$, where $b = 1 - 2a$. If $(3b)^{i(n)}(3a)^{n-i(n)}$ converge as n tends to infinity, the function F_a is differentiable at x. Otherwise, it is not.

If $\xi_j = 1$ occur only for a finite number of j's, $|f_n'(x)|$ obviously tends to infinity. If $\xi_j = 1$ occur infinitely often, then (since $b < 0$) the sequence $f_n'(x)$ changes its sign infinitely often. It therefore converges if and only if it converges to zero.

Let $\gamma \in [0, 1]$ be defined by

$$\liminf_{n\to\infty} \frac{i(n)}{n} = \gamma. \tag{2}$$

Then, $(3b)^{i(n)}(3a)^{n-i(n)} \to 0$ if $|3b^\gamma a^{1-\gamma}| < 1$. Consequently, $F_a'(x) = 0$, if

$$\gamma > \frac{-\log(3a)}{\log(2a-1) - \log a}. \tag{3}$$

In particuler, if $\xi_j = 1$ for all j except for a finite number, we then have $\gamma = 1$ and accordingly $F_a'(x) = 0$. Thus $F_a'(x) = 0$ if x is the mid-point of $k/3^n$ and $(k+1)/3^n$.

On the other hand, if $\gamma < -\log(3a)/[\log(2a-1) - \log a]$, then $(3b)^{i(n)}(3a)^{n-i(n)}$ diverges. □

The graph of $\phi(a) = -\log(3a)/[\log(2a-1) - \log a]$ is shown in Fig. 3. It decreases monotonically from 1 to zero, as a decreases from 2/3 to 1/2.

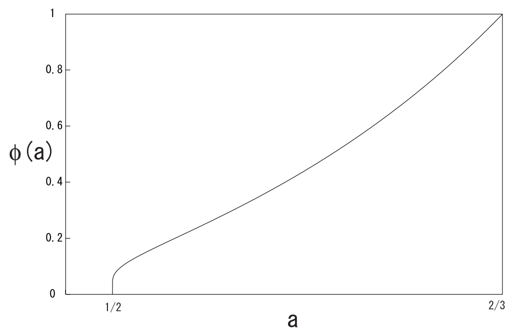

Fig. 3. The graph of $\phi(a)$.

Remark 1. Suppose now that $0 < a < 1/3$. In this case F_a' vanishes at $x = k/3^n$ and $F_a'(k/3^n + 1/(2\cdot 3^n)) = +\infty$. Also, the function has zero derivative if

$$\gamma < \frac{-\log(3a)}{\log(1-2a) - \log a}.$$

Since $0 < 3b < 1 < 3a$, there exists infinity of x, at which $(3b)^{i(n)}(3a)^{n-i(n)}$ converges to a nonzero limit. The author, however, does not know a simple characterization of them.

We finally consider the following question. Let the set of all the points at which F_a is nondifferentiable be denoted by S_a. Then $|S_a|$, the Lebesgue measure of S_a, is zero for $a \leq 1/2$, and is one for $a \geq 2/3$. What can we say about $|S_a|$ for $1/2 < a < 2/3$?

The (incomplete) answer is as follows:

Theorem 4. *Let a_0 be the unique root of $54a^3 - 27a^2 = 1$ in $1/2 < a < 2/3$. Then $|S_a| = 0$ if $a < a_0$, and $|S_a| = 1$ if $a > a_0$.*

Proof. Note first that a_0 is the root of

$$\frac{-\log(3a)}{\log(2a-1) - \log a} = \frac{1}{3}.$$

Since it is elementary, we omit the proof of the fact that this equation has a unique root in $1/2 < a < 2/3$.

Suppose now that $a < a_0$. We can take a γ_0 such that $\frac{-\log(3a)}{\log(2a-1)-\log a} < \gamma_0 < 1/3$. We then consider the set of all the real numbers for which

$$\liminf_{n\to\infty} \frac{i(n)}{n} \geq \gamma_0.$$

We see that the set is included in S_a. Therefore the proof of the first half is complete if we have shown that the measure of the set is equal to one. The measure is actually equal to

$$\lim_{n\to\infty} \sum_{k=[\gamma_0 n]}^{n} \binom{n}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k},$$

where $[\gamma_0 n]$ denotes the largest integer not exceeding $\gamma_0 n$. This is equal, by the central limit theorem, to

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\theta_n}^{\infty} \exp(-z^2)\mathrm{d}z,$$

where

$$\theta_n = \left([\gamma_0 n] - \frac{n}{3} + \frac{1}{2}\right) \Big/ \left(\sqrt{n}\frac{2}{9}\right).$$

Since $\gamma_0 < 1/3$, the limit is equal to one. By quite an analogous way we can prove the latter half. □

A numerical computation shows that $a_0 \approx 0.5592$. The author does not know about $|S_{a_0}|$.

3. Concluding remarks. If we introduce another parameter a' in such a way that a piecewise affine function, obtained by joining $(0,0)$–$(1/3,a)$–$(2/3,a')$–$(1,1)$, then we obtain new functions of more variety. Further, 1/3 and2/3 may be replaced by other numbers. The simplicity, however, seems to be best seen in our present construction.

Prof. J. Kigami pointed out that the construction of functions by de Rham [3] has a similarity to ours. The similarity is indeed striking. But none of his functions belongs to our class, and none of ours is his. He also pointed out that our construction with $a = 1$ has a similarity to Moore [7], which constructed a continuous nowhere differentiable function by our operation with $a = 1$ and an additional construction.

Although we cannot see a direct connection, it may be helpful if we refer some non-differentiable functions expressed by binary and multi-nary expansions. Kawamura [5] generalized the results of [3, 4], and obtained, among others, a new class of continuous, nowhere differentiable functions by means of certain functional equations. Bush [2] constructed continuous, nowhere differentiable functions by using m-nary expansions of the independent variable, where m is any positive integer > 2. Swift [11] defined a one by means of the ternary expansion of the independent variable.

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References

[1] N. Bourbaki, *Functions of a real variable*, Translated from the 1976 French original by Philip Spain, Springer, Berlin, 2004, p.35, Problem 1–2.

[2] K. A. Bush, Continuous functions without derivatives, Amer. Math. Monthly **59** (1952), 222–225.

[3] G. de Rham, Sur quelques courbes definies par des equations fonctionnelles, Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956/1957), 101–113.

[4] M. Hata and M. Yamaguti, The Takagi function and its generalization, Japan J. Appl. Math. **1** (1984), no. 1, 183–199.

[5] K. Kawamura, On the classification of self-similar sets determined by two contractions on the plane, J. Math. Kyoto Univ. **42** (2002), no. 2, 255–286.

[6] A. B. Kharazishvili, *Strange functions in real analysis*, Dekker, New York, 2000.

[7] E. H. Moore, On certain crinkly curves, Trans. Amer. Math. Soc. **1** (1900), no. 1, 72–90.

[8] S. Russ, Bolzano's analytic programme, Math. Intelligencer **14** (1992), no. 3, 45–53.

[9] F. W. Perkins, An elementary example of a continuous non-differentiable functions, Amer. Math. Monthly **34** (1927), 476–478.

[10] A. N. Singh, *The theory and construction of non-differentiable functions*, in *Squaring the Circle and Other Monographs*, Chelsea, New York, 1953.

[11] W. C. Swift, Simple constructions of nondifferentiable functions and space-filling curves, Amer. Math. Month. **68** (1961), 653–655.

[12] M. Yamaguti, M. Hata and J. Kigami, *Mathematics of fractals*, Translated from the 1993 Japanese original by Kiki Hudson, Amer. Math. Soc., Providence, RI, 1997.