An example of nonuniqueness of the Cauchy problem for the Hermite heat equation

By Bishnu P. Dhungana*), **)

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Abstract: Using Mehler kernel, we give an example of nontrivial solution of the homogeneous Cauchy problem of the Hermite heat equation, which is, for each t, bounded in the space variables.

Key words: Hermite function; Mehler kernel; Hermite heat equation.

1. Introduction. For $k = 0, 1, 2, \dots$, we denote by h_k the normalized Hermite functions on $\mathbf{R}$ defined by

$$h_k(x) = (2^k k! \sqrt{\pi})^{-\frac{1}{2}} \tilde{h}_k(x)$$

where $\tilde{h}_k$ is the Hermite function on $\mathbf{R}$ defined by

$$\tilde{h}_k(x) = (-1)^k e^{\frac{1}{2}x^2} \frac{d^k}{dx^k} e^{-x^2}.$$

For all $x, \xi \in \mathbf{R}$ and $w \in \mathbf{C}$ with $|w| < 1$, the well known Mehler formula (p. 107, [6]) is

$$(1.1) \quad \sum_{k=0}^{\infty} \frac{\tilde{h}_k(x)\tilde{h}_k(\xi)}{2^k k!} w^k = (1-w^2)^{-\frac{1}{2}} e^{-\frac{1}{2}\frac{1+w^2}{1-w^2}(x^2+\xi^2) + \frac{2w}{1-w^2}x\xi}$$

where the series is uniformly and absolutely convergent on $\{w \in \mathbf{C} : |w| < 1\}$. In view of (1.1), it is easy to see that

$$\sum_{k=0}^{\infty} e^{-(2k+1)t} h_k(x) h_k(\xi) = \frac{e^{-t}}{\sqrt{\pi}(1-e^{-4t})^{\frac{1}{2}}} e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}(x-\xi)^2 - \frac{1-e^{-2t}}{1+e^{-2t}}x\xi}$$

for $x, \xi \in \mathbf{R}$ and $t > 0$. We denote by $E(x, \xi, t)$ the Mehler kernel and define by

$$(1.2) \quad E(x,\xi,t) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} e^{-(2k+1)t} h_k(x) h_k(\xi), & t > 0 \\ 0, & t \le 0. \end{cases}$$

For fixed $\xi \in \mathbf{R}$, we easily see that $E(x, \xi, t)$ satisfies the Hermite heat equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x^2\right) U(x,t) = 0, \quad x \in \mathbf{R},\ 0 < t < \infty.$$

Moreover for each $x \in \mathbf{R}$ and each $t > 0$, $E(x, \zeta, t)$ is an entire function of $\zeta \in \mathbf{C}$.

As a particular case of the second order parabolic equation, the following is the famous uniqueness theorem on the Cauchy problem of the Hermite heat equation:

Theorem 1.1 (p. 86, [1]). *Let $T > 0$ and $U(x,t)$ be a continuous function in $\mathbf{R} \times [0,T]$ such that*

(a) $\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x^2\right) U(x,t) = 0$ *in* $\mathbf{R} \times (0,T)$,

(b) *for some constants* C, $A > 0$,

$$|U(x,t)| \le C\ e^{A(x^2+1)} \quad in\ \ \mathbf{R} \times (0,T),$$

(c) $U(x,0) = 0$ *for* $x \in \mathbf{R}$.

Then $U(x,t) \equiv 0$ *in* $\mathbf{R} \times [0,T]$.

The aim of this paper is to give an example of nonuniqueness for the Cauchy problem of the Hermite heat equation, which is, for each t, uniformly bounded in the x variable.

2. Main results.

Lemma 2.1. *Let $E(x, \xi, t)$ be the Mehler kernel as defined in* (1.2) *and let $0 < \epsilon < 1$. For each $M > 0$, let $L_\epsilon = \{\xi + i\eta : \xi > M - \epsilon,\ |\eta| \le \epsilon\}$. Then, for $t > 0$, there exist some constants $C_1, C_2 > 0$ such that*

$$\sup_{\zeta \in L_\epsilon} |E(x,\zeta,t)| \le C_2 \exp\left(\frac{\epsilon}{t} - \frac{C_1}{2t} d(x, L_\epsilon)^2\right).$$

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*) Department of Mathematics, Mahendra Ratna Campus Tribhuvan University, Kathmandu, Nepal.

**) Department of Mathematics, Meijo University, 1-501, Shiogamaguchi, Tempaku-ku, Nagoya, Aichi 468-8502.

Proof. For $\zeta = \xi + i\eta$ and $t > 0$, we have

$$|E(x,\zeta,t)| = \frac{e^{-t}\, e^{-\left\{\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}(x-\xi)^2 + \frac{1-e^{-2t}}{1+e^{-2t}}x\xi\right\} + \frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}\eta^2}}{\sqrt{\pi(1-e^{-4t})}}.$$

Since $\frac{e^{-t}}{\sqrt{1-e^{-4t}}} \le \frac{1}{2\sqrt{t}}$ for every $t > 0$, we have

$$|E(x,\zeta,t)| \le \frac{e^{-\left\{\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}(x-\xi)^2 + \frac{1-e^{-2t}}{1+e^{-2t}}x\xi\right\} + \frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}\eta^2}}{2\sqrt{\pi t}}.$$

Put $P = \frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}$ and $Q = \frac{1-e^{-2t}}{1+e^{-2t}}$ for positive t. Then $P > 0$ and $Q > 0$. Using the inequality

$$P(x-\xi)^2 + Qx\xi \ge \left(P - \frac{Q}{2}\right)(x-\xi)^2,$$

we obtain that

$$|E(x,\zeta,t)| \le \frac{1}{2\sqrt{\pi t}} e^{\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}\eta^2 - \frac{e^{-2t}}{1-e^{-4t}}\left\{\eta^2 + (x-\xi)^2\right\}}.$$

Since $d(x, L_\epsilon)^2 \le \eta^2 + (x-\xi)^2$ for every $\zeta \in L_\epsilon$, we have

(2.1)
$$\sup_{\zeta\in L_\epsilon} |E(x,\zeta,t)| \le \frac{1}{2\sqrt{\pi t}} e^{\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}\epsilon^2 - \frac{e^{-2t}}{1-e^{-4t}}d(x,L_\epsilon)^2}.$$

For $0 < t < 1$, it is not difficult to see that

$$\frac{1+e^{-2t}}{2(1-e^{-2t})} \le \frac{1}{t}, \qquad \frac{e^{-2t}}{1-e^{-4t}} \ge \frac{C_1}{t}$$

for some constant $C_1 > 0$. It then follows from (2.1) that for $0 < t < 1$

$$\sup_{\zeta\in L_\epsilon} |E(x,\zeta,t)| \le \frac{1}{2\sqrt{\pi t}} e^{\frac{\epsilon^2}{t} - \frac{C_1}{t}d(x,L_\epsilon)^2}. \tag{2.2}$$

But for $0 < \epsilon < 1$ and $0 < t < 1$, there exists some constant $C_2 > 0$ such that

$$\frac{1}{2\sqrt{\pi t}} e^{\frac{\epsilon^2}{t} - \frac{C_1}{t}d(x,L_\epsilon)^2 - \frac{\epsilon}{t} + \frac{C_1}{2t}d(x,L_\epsilon)^2} \le C_2. \tag{2.3}$$

It then follows from (2.2) and (2.3) that for $0 < t < 1$

(2.4)
$$\sup_{\zeta\in L_\epsilon} |E(x,\zeta,t)| \le C_2 \exp\left(\frac{\epsilon}{t} - \frac{C_1}{2t}d(x,L_\epsilon)^2\right).$$

But from (2.1) we see that

$$\sup_{\zeta\in L_\epsilon} |E(x,\zeta,t)| \to 0 \quad \text{as} \quad t \to \infty. \tag{2.5}$$

Hence in view of (2.4) and (2.5), we obtain

$$\sup_{\zeta\in L_\epsilon} |E(x,\zeta,t)| \le C_2 \exp\left(\frac{\epsilon}{t} - \frac{C_1}{2t}d(x,L_\epsilon)^2\right)$$

for $t > 0$ and $0 < \epsilon < 1$. □

Theorem 2.1. *Let $T > 0$ be fixed and $\epsilon > 0$ be arbitrary. Then there exists a continuous function $U(x,t)$ on $\mathbf{R} \times [0,T]$ satisfying*

(2.6)
$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x^2\right)U(x,t) = 0 \quad \textit{in} \quad \mathbf{R}\times(0,T),$$

for some constant $C := C(\epsilon)$

$$|U(x,t)| \le C\exp\left(\frac{\epsilon}{t}\right) \quad \textit{in} \quad \mathbf{R}\times(0,T), \tag{2.7}$$

$$U(x,0) = 0 \quad \textit{for} \quad x \in \mathbf{R}. \tag{2.8}$$

But $U(x,t)$ is not identically zero in $\mathbf{R}\times[0,T]$.

Proof. For each $M > 0$, consider a curve

$$\gamma_M = \gamma_1 \cup \gamma_2 \cup \gamma_3$$

in the complex plane $\mathbf{C}$ where $\zeta = \xi + i\eta$ and

$$\gamma_1 = \left\{\zeta \in \mathbf{C} | \xi\eta = \frac{\pi}{2}, \quad \xi \ge M\right\},$$
$$\gamma_2 = \left\{\zeta \in \mathbf{C} | \xi = M, \quad |\eta| \le \frac{\pi}{2M}\right\},$$
$$\gamma_3 = \left\{\zeta \in \mathbf{C} | \xi\eta = -\frac{\pi}{2}, \quad \xi \ge M\right\}.$$

Define a function $U(x,t)$ on $\mathbf{R}\times(0,T)$ by

$$U(x,t) = \frac{1}{2\pi i}\int_{\gamma_M} E(x,\zeta,t)\exp(e^{\zeta^2})d\zeta \tag{2.9}$$

where the integral is taken counterclockwise. Since

$$|\exp(e^{\zeta^2})| = \exp(-e^{\xi^2-\eta^2})$$

on the curve $\xi\eta = \pm\frac{\pi}{2}$, it shows that the function $\exp(e^{\zeta^2})$ decreases very rapidly as $\xi \to \infty$ on the curve $\xi\eta = \pm\frac{\pi}{2}$. So $U(x,t)$ is well defined on $\mathbf{R}\times(0,T)$. Also the integral is independent of $M > 0$ and moreover it satisfies (2.6).

For $0 < \epsilon < 1$, let L_ϵ be as in Lemma 2.1. Choose $M > 0$ sufficiently large so that $\gamma_M \subset L_\epsilon$. Since the integral

$$\frac{1}{2\pi}\int_{\gamma_M} |\exp(e^{\zeta^2})|\,|d\zeta|$$

is finite, we obtain

$$|U(x,t)| \le C_M \sup_{\zeta\in L_\epsilon}|E(x,\zeta,t)| \tag{2.10}$$

for some constant $C_M > 0$. By (2.10) and Lemma 2.1, we have

(2.11)
$$|U(x,t)| \le C\exp\left(\frac{\epsilon}{t} - \frac{C_1}{2t}d(x,L_\epsilon)^2\right), \quad t > 0$$

for some constants $C, C_1 > 0$. Moreover the inequality (2.11) holds for any $\epsilon > 0$ and we easily obtain (2.7) from (2.11).

Since the integral (2.9) is independent of M, (2.11) means that for any $R > 0$,

$$\sup_{x \le R} |U(x,t)| \to 0 \quad \text{as} \quad t \to 0_+ \tag{2.12}$$

which implies that $U(x,t)$ is continuous on $\mathbf{R} \times [0,T)$ and $U(x,0) = 0$ on $\mathbf{R}$.

Now we show that $U(x,t)$ is not identically zero. Since

$$E_x(x,\zeta,t) = \frac{e^{-t}\, e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}(x-\zeta)^2 - \frac{1-e^{-2t}}{1+e^{-2t}}x\zeta}}{\sqrt{\pi(1-e^{-4t})}\left(-\frac{1+e^{-4t}}{1-e^{-4t}}(x-\zeta) - \frac{1-e^{-2t}}{1+e^{-2t}}\zeta\right)^{-1}}$$

we obtain that for suficiently large $M > 0$,

$$\begin{aligned} U_x\left(0, -\frac{1}{4}\log\frac{1}{3}\right) &= \frac{3^{\frac{3}{4}}}{\sqrt{2\pi}\ 2\pi i}\int_{\gamma_M} \zeta\, e^{e^{\zeta^2} - \zeta^2} d\zeta \\ &= \frac{3^{\frac{3}{4}}}{\sqrt{8\pi}\ 2\pi i}\int_{L_\pi} e^{-\zeta} e^{e^{\zeta}} d\zeta \\ &= -\frac{3^{\frac{3}{4}}}{\sqrt{8\pi}\ \Gamma(2)} \neq 0 \end{aligned}$$

where the last equality follows from the Hankel integral formula for Γ functions (p. 245, [5]). This completes the proof. □

Remark 2.1. Rauch in his monograph [4] proved that

$$u(x,t) = \int_{\Gamma_0} e^{x(-z)^{1/2}} e^{-z^\alpha} e^{zt} dz, \quad \alpha \in (1/2, 1)$$

where Γ_0 denote the contour $\mathrm{Re}(z) = a \ge 0$ oriented in the direction of the increasing imaginary part, is a nontrivial solution of the heat equation vanishing identically for $t < 0$. But for each t, it is not bounded in the space variable. A better nontrivial solution of the heat equation with null initial data was presented in [3] via heat kernel. Quite naturally, our effort to find an example of nonuniqueness of the Cauchy problem for the Hermite heat equation was a challenging task. We fulfilled it with the use of Mehler kernel. Though the example is based on the techniques involved in [3], the role of Mehler kernel that gives a different mode to the example is considered to be predominant. In particular, the solution converges [see Section 2, (2.12)] to zero uniformly for $x \in \{|x| \le R : R > 0\}$ as $t \to 0$ which is an important aspect of the example. Moreover, extending the definition of $U(x,t)$ for $t < 0$ by $U \equiv 0$ we have $U(x,t) \in \mathcal{C}^\infty(\mathbf{R}_x \times \mathbf{R}_t)$ by virtue of (2.12) and hypoelliptic property of the Hermite heat operator. That is to say, $U(x,t)$ is a nontrivial $\mathcal{C}^\infty$-solution satisfying the Hermite heat equation and vanishing identically for $t < 0$. Furthermore possibly the first example of nonuniqueness of the Cauchy problem for the Hermite heat equation, it might be useful for further researches in partial differential equations. For example, it remains to study about the optimal growth condition with respect to t, for small $t > 0$, for the uniqueness of the solution to the Cauchy problem for the Hermite heat equation (cf. [2]).

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