Exterior differential algebras and flat connections on Weyl groups

By Anatol N. Kirillov*) Toshiaki Maeno**)

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Abstract: We study some aspects of noncommutative differential geometry on a finite Weyl group in the sense of S. Woronowicz, K. Bresser *et al.*, and S. Majid. For any finite Weyl group W we consider the subalgebra generated by flat connections in the left-invariant exterior differential algebra of W.

For root systems of type A and D we describe a set of relations between the flat connections, which conjecturally is a complete set.

Key words: Noncommutative differential calculus; Weyl groups.

Introduction. The study of higher order differential structures on Hopf algebras was initiated by S. L. Woronowicz [6], and further developed by K. Bresser *et al.* [1] and S. Majid [5] for algebras of functions on finite groups. In particular, S. Majid has introduced and studied flat connections on the symmetric group S_N. In our paper, we study the algebra generated by flat connections in a sense of Majid on a finite Weyl group. This is an interesting problem which is not treated in [5].

We consider the differential structure with respect to the set of reflections. Since the complete set of the defining relations of the left-invariant exterior differential algebra has not yet been determined in general, we will work on its quadratic version Λ_{quad} for the root system of type A or D, and on its quartic version Λ_{quar} for the root system of type B. Our main result describes a set of relations among flat connections on Weyl groups of type A and D. Conjecturally, these relations are complete set of relations among flat connections in Λ_{quad}. We expect some connections of our construction with Schubert calculus on flag varieties [4].

1. Woronowicz exterior algebra.

Woronowicz exterior algebra was introduced in [6] for the study of higher order differential structure on the quantum groups. In the category of modules over a commutative algebra, the exterior products of a module are constructed by using the canonical action of the symmetric groups on the tensor products. In general, such a construction does not work in the category of bimodules over a noncommutative algebra because of lack of canonical action of the symmetric groups on the tensor products. However, in the category of bimodules over a Hopf algebra, one can obtain a natural generalization of the exterior product. In this paper, all (Hopf) algebras are over a field K of characteristic zero. Let H be a Hopf algebra.

Definition 1.1. A bimodule M over H is called a left (resp. right) covariant bimolude if M has a left (resp. right) H-comodule structure compatible with the bimodule structure. A bimodule M is called a bicovariant bimodule if M has left and right covariant bimodule structures and the left coaction and the right coaction commute.

Definition 1.2. Let M be a bicovariant bimodule over a Hopf algebra H. An element $x \in M$ is called left (resp. right) invariant if x is mapped to $1 \otimes x$ (resp. $x \otimes 1$) by the comodule action of H.

Lemma 1.1. *There exists a unique braiding* $\Psi : M \otimes_H M \to M \otimes_H M$ *such that* $\Psi(\omega \otimes \eta) = \eta \otimes \omega$ *for left invariant* ω *and right invariant* η.

The homomorphism Ψ induces a homomorphism $\Psi_i : M^{\otimes_H n} \to M^{\otimes_H n}$, $1 \le i \le n$, which acts as Ψ on i-th and $(i+1)$-st components and acts identically on the other components. Take an element $w \in S_n$ and its reduced decomposition $w = s_{i_1} \cdots s_{i_l}$, $s_i = (i, i+1)$. Then we can associate a homomorphism $\Psi(w) : M^{\otimes_H n} \to M^{\otimes_H n}$ to the element w by defining $\Psi(w) = \Psi_{i_1} \cdots \Psi_{i_l}$. Since Ψ_i's satisfy the braid relations, the homomorphism $\Psi(w)$ is independent of the choice of reduced decomposi-

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*) Research Institute for Mathematical Sciences, Kyoto University, Kita-Shirakawa-Oiwake-cho, Sakyo-ku, Kyoto 606-8502.

**) Department of Mathematics, Kyoto University, Kita-Shirakawa-Oiwake-cho, Sakyo-ku, Kyoto 606-8502.

tion of w. Now we define the antisymmetrizer A_n on $M^{\otimes_H n}$ by the formula $A_n = \sum_{w\in S_n} \mathrm{sgn}(w)\Psi(w)$.

Definition 1.3. Woronowicz exterior algebra $\bigwedge M$ is a quotient of the tensor algebra of M over H by the kernel of the antisymmetrizer, i.e.

$$\bigwedge M := T_H M / \bigoplus_n \mathrm{Ker}(A_n).$$

2. Differential structure on the Weyl group.

2.1. Differential structure on a finite group. First of all, let us remind some fundamental facts on noncommutative differential structures on the finite group following [1] and [5].

Definition 2.1. Let A be a K-algebra. The first order differential structure of A is a pair of A-bimodule Ω^1_A and a K-linear map $d : A \to \Omega^1_A$ such that the map d satisfies the Leibniz rule $d(ab) = (da)b + a(db)$, for $a, b \in A$, and the image of d generates Ω^1_A as a left A-module.

Definition 2.2. Let H be a Hopf algebra. The first order differential structure $(d : H \to \Omega^1_H)$ is said to be bicovariant if Ω^1_H has a structure of a bicovariant bimodule and the map d is a bicomodule homomorphism.

As a consequence of the construction in Section 1, we have the Woronowicz exterior algebra of a bicovariant differential structure Ω^1_H of a Hopf algebra H.

Definition 2.3. The Woronowicz exterior differential algebra Ω_w for the bicovariant differential structure of a Hopf algebra H is a Woronowicz exterior algebra of Ω^1_H, i.e. $\Omega_w := \bigwedge \Omega^1_H$. The left invariant subalgebra of Ω_w is denoted by Λ_w.

Let G be a finite group and H an algebra of functions on G taking values on K. Now we consider the differential structure on the Hopf algebra $H = K(G)$. The set of the delta functions $\{\delta_g \mid g \in G\}$ can be taken as a linear basis of H.

Now we construct a canonical differential structure of the algebra H. Take a subset $\mathcal{C}$ of G which does not contain the identity element. Let $D_{\mathcal{C}} = \{(x, y) \in G \times G \mid x^{-1}y \in \mathcal{C}\}$. Define $\Omega^1(G)$ as a K-linear space generated by the set $\{\delta_x \otimes \delta_y \mid (x, y) \in D_{\mathcal{C}}\}$, and

$$df = \sum_{(x,y)\in D_{\mathcal{C}}} (f(y) - f(x))\delta_x \otimes \delta_y, \quad \text{for } f \in H.$$

Then $(d : H \to \Omega^1(G))$ is a first order differential structure on H. All left covariant differential structures on H are of this form, and $\Omega^1(G)$ is bicovariant if and only if the set $\mathcal{C}$ is stable under the adjoint action of G. Hence, simple bicovariant differential structures on H are classified by nontrivial conjugacy classes of G.

For an element $a \in G$, let $e_a = \sum_{g\in G} \delta_g d\delta_{ga}$. Then the left invariant subalgebra Λ_w is a K-linear subspace spanned by e_a, $a \in \mathcal{C}$.

2.2. Differential structure on the Weyl group. Now we assume the group W to be a Weyl group. Denote by Δ the set roots, and Δ_+ the set of positive roots. As we have seen in the previous subsection, bicovariant differential structures on $H = K(W)$ are corresponding to adjoint invariant subsets of W. We take $\mathcal{C} = \mathcal{C}_{refl}$ the set of reflections as the simplest adjoint invariant subset of W.

Remark 2.1. For a simply-laced root system, the set $\mathcal{C}$ forms a conjugacy class. However, for a nonsimply-laced system the set $\mathcal{C}$ splits into a disjoint union of two conjugacy classes: $\mathcal{C} = \mathcal{C}_l \cup \mathcal{C}_s$, where $\mathcal{C}_l$ (resp. $\mathcal{C}_s$) is the set of reflections with respect to the long (resp. short) roots. We can see that $\Lambda_w(B_n; \mathcal{C}_l) \cong \Lambda_w(C_n; \mathcal{C}_s) \cong \Lambda_w(D_n; \mathcal{C})$,

$$\Lambda_w(B_n; \mathcal{C}_s) \cong \Lambda_w(C_n; \mathcal{C}_l) \cong \Lambda_w((A_1)^n; \mathcal{C}),$$

$$\Lambda_w(G_2; \mathcal{C}_l) \cong \Lambda_w(G_2; \mathcal{C}_s) \cong \Lambda_w(A_2; \mathcal{C}).$$

This fact shows that the *simple* differential structure corresponding to $\mathcal{C}_l$ or $\mathcal{C}_s$ is not appropriate to investigate the differential structure for nonsimply-laced root systems. For that reason, we consider the differential structure obtained from the set $\mathcal{C}$, which is not simple differential structure for nonsimply-laced root system. The algebra $\Lambda_w(X, \mathcal{C})$ will be denoted simply by $\Lambda_w(X)$.

We define a quadratic version of the left-invariant differential algebra follwing [5]: $\Lambda_{quad} := T_K\Lambda^1 / \ker(1 - \Psi)$.

Conjecture 2.1. *For simply-laced root systems, $\Lambda \cong \Lambda_{quad}$. For the root system of type A this conjecture was stated by S. Majid* [5].

Remark 2.2. For nonsimply-laced root systems, Λ_{quad} is not isomorphic to Λ_w.

Example 2.1. The algebra $\Lambda_{quad}(B_n)$ is generated by $e_{(ij)}$, $e_{\overline{(ij)}}$ and $e_{(i)}$, where (ij), $\overline{(ij)}$ and (i) are reflections. The defining quadratic relations are:

$$e_{(ij)}^2 = e_{\overline{(ij)}}^2 = e_{(i)}^2 = 0,$$

$$e_{(ij)}e_{(kl)} + e_{(kl)}e_{(ij)} = e_{(ij)}e_{\overline{(kl)}} + e_{\overline{(kl)}}e_{(ij)}$$

$$= e_{\overline{(ij)}}e_{\overline{(kl)}} + e_{\overline{(kl)}}e_{\overline{(ij)}} = 0, \text{ for } \{i, j\} \cap \{k, l\} = \emptyset,$$

$$e_{(i)}e_{(j)} + e_{(j)}e_{(i)} = e_{(ij)}e_{\overline{(ij)}} + e_{\overline{(ij)}}e_{(ij)}$$
$$= e_{(ij)}e_{(k)} + e_{(k)}e_{(ij)} = e_{\overline{(ij)}}e_{(k)} + e_{(k)}e_{\overline{(ij)}},$$
$$\text{if } k \neq i, j,$$
$$e_{(ij)}e_{(jk)} + e_{(jk)}e_{(ki)} + e_{(ki)}e_{(ij)} = 0,$$
$$e_{\overline{(ik)}}e_{(ij)} + e_{(ji)}e_{\overline{(jk)}} + e_{\overline{(kj)}}e_{\overline{(ik)}} = 0,$$
$$e_{(ij)}e_{(i)} + e_{(j)}e_{(ij)} + e_{(i)}e_{\overline{(ij)}} + e_{\overline{(ij)}}e_{(j)} = 0.$$

The algebra $\Lambda_{quad}(D_n)$ is a quotient of $\Lambda_{quad}(B_n)$.

Remark 2.3. The algebra $\Lambda_{quad}(B_n)$ is not isomorphic to $\Lambda_w(B_n)$. For example, the relations

$$e_{\overline{(ij)}}e_{(i)}e_{(ij)}e_{(i)} + e_{(i)}e_{(ij)}e_{(i)}e_{\overline{(ij)}}$$
$$+ e_{(ij)}e_{(i)}e_{\overline{(ij)}}e_{(i)} + e_{(i)}e_{\overline{(ij)}}e_{(i)}e_{(ij)} = 0,$$
$$e_{(ij)}e_{(i)}e_{(ij)}e_{(i)} + e_{(i)}e_{(ij)}e_{(i)}e_{(ij)} = 0$$

hold in $\Lambda_w(B_n)$, but they do not in $\Lambda_{quad}(B_n)$. We denote by $\Lambda_{quar}(B_n)$ the quotient algebra of $\Lambda_{quad}(B_n)$ by the ideal generated by the quartic relations above.

Example 2.2. The algebra $\Lambda_{quad}(B_2)$ is infinite dimensional. It has the Hilbert polynomial $(1+t)^2(1-t)^{-2}$. In the algebra $\Lambda_w(B_2)$, the quartic relations

$$e_{\overline{(12)}}e_{(1)}e_{(12)}e_{(1)} + e_{(1)}e_{(12)}e_{(1)}e_{\overline{(12)}}$$
$$+ e_{(12)}e_{(1)}e_{\overline{(12)}}e_{(1)} + e_{(1)}e_{\overline{(12)}}e_{(1)}e_{(12)} = 0$$

and $e_{(12)}e_{(1)}e_{(12)}e_{(1)} + e_{(1)}e_{(12)}e_{(1)}e_{(12)} = 0$ hold. The algebra $\Lambda_{quar}(B_2)$ obtained by adding the quartic relations above to $\Lambda_{quad}(B_2)$ is finite dimensional and has the Hilbert polynomial $(1+t)^4(1+t^2)^2$. In particular, $\Lambda_w(B_2)$ is finite dimensional. The anticommutative quotient of the algebra $\Lambda_{quad}(B_2)$ has the Hilbert polynomial $1+4t+5t^2+2t^3 = (1+t)^2(1+2t)$.

Conjecture 2.2. *The relations in Example* 2.1 *and Remark* 2.3 *are the complete set of relations for $\Lambda_w(B_n)$, i.e. $\Lambda_w(B_n) \cong \Lambda_{quar}(B_n)$.*

3. $U(1)$-gauge theory.

The algebra Λ_w has a structure of a differential graded algebra over K. Denote by $H^*(W)$ the cohomology group of the differential graded algebra Λ_w. Let $\theta = \sum_{a\in\mathcal{C}} e_a$. If W is nonsimply-laced, we define $\theta_1 = \sum_{a\in\mathcal{C}_l} e_a$ and $\theta_2 = \sum_{a\in\mathcal{C}_s} e_a$.

Proposition 3.1. *For simply-laced root system, $H^1(W) = K\cdot\theta$. For nonsimply-laced root system, $H^1(W) = K\cdot\theta_1 \oplus K\cdot\theta_2$.*

Proof. Since $d\delta_g = \sum_{c\in\mathcal{C}}(\delta_{gc} - \delta_g)e_c$ and $e_c \cdot \delta_g = \delta_{gc}\cdot e_c$, we have $de_a = \sum_{g\in W} d\delta_g d\delta_{ga} = \theta e_a + e_a\theta$. We can show that if $\eta = \sum_{a\in\mathcal{C}} \eta_a e_a$ is a closed 1-form, then $\eta_a = \eta_b$ must be satisfied when a and b are conjugate each other. □

Let Ω^1_H be a bicovariant differential structure of a Hopf algebra H.

Definition 3.1. For a 1-form $\eta \in \Omega^1_H$, the covariant curvature is defined by $F(\eta) = d\eta + \eta\wedge\eta$. If $F(\eta) = 0$, η is called a flat $U(1)$-connection.

As we have seen in Remark 2.1, the simple differential structures of nonsimply-laced Weyl groups are reduced to the ones of simply-laced Weyl groups. Hence, we restrict our considerations to the case of simply-laced root system $X = (\Delta, W)$. Fix the set of simple roots $\Sigma \subset \Delta$. Let ω_α be a fundamental dominant dominant weight corresponding to a simple root $\alpha\in\Sigma$. Denote by $(\nu_\alpha)_\alpha$ Schmidt's orthogonalization of $(\omega_\alpha)_\alpha$. We define the 1-forms θ^X_α for $\alpha\in\Sigma$ by $\theta^X_\alpha := \sum_{\gamma\in\Delta_+(\alpha)}\langle\nu_\alpha, \gamma^\vee\rangle e_{s_\gamma}$, where $\Delta_+(\alpha)$ is the set of the roots γ satisfying the condition $\langle\nu_\alpha, \gamma\rangle > 0$.

Proposition 3.2. *For the classical root systems the 1-forms $\theta_\alpha = \theta^X_\alpha$ satisfy the relations of anticommutativity $\theta_\alpha\theta_\beta + \theta_\beta\theta_\alpha = 0$ and flatness relations $F(-\theta_\alpha) = 0$.*

Proof. This can be shown by direct computations. (See Section 5). □

Remark 3.1. The 1-forms θ_α can be considered as an analogue of the Dunkl elements introduced in [2] and [4].

Example 3.1. Here we give an example in the exceptional root system of type G_2. Let α be the short simple root and β the long one. Then the set of positive roots is $\Delta_+ = \{a_1 = \alpha, a_2 = 3\alpha+\beta, a_3 = 3\alpha+2\beta, a_4 = 2\alpha+\beta, a_5 = \alpha+\beta, a_6 = \beta\}$. Let s_i be the reflection with respect to a_i and $e_i := e_{s_i}$. Then the relations $e_i^2 = 0$,

$$e_1e_4 + e_4e_1 = e_2e_5 + e_5e_2 = e_3e_6 + e_6e_3 = 0,$$
$$e_1e_3 + e_3e_5 + e_5e_1 = e_3e_1 + e_5e_3 + e_1e_5 = 0,$$
$$e_2e_4 + e_4e_6 + e_6e_2 = e_4e_2 + e_6e_4 + e_2e_6 = 0,$$
$$e_1e_2 + e_2e_3 + e_3e_4 + e_4e_5 + e_5e_6 + e_6e_1 = 0,$$
$$e_2e_1 + e_3e_2 + e_4e_3 + e_5e_4 + e_6e_5 + e_1e_6 = 0$$

hold in $\Lambda_w(G_2)$. The first cohomology group is

$$H^1(W(G_2)) = K\cdot(e_1+e_3+e_5) + K\cdot(e_2+e_4+e_6).$$

Moreover, the 1-forms $\eta_1 = -(2e_1+e_2+e_3+e_5+e_6)$ and $\eta_2 = -(e_2+e_3+2e_4+e_5+e_6)$ define flat connections which satisfy the anticommutativity $\eta_1\eta_2 + \eta_2\eta_1 = 0$.

4. Hopf algebra structure. We introduce a Hopf algebra structure on $K\langle W\rangle \otimes_K \Lambda_{quad}(W)$. We consider $K\langle W\rangle \otimes_K \Lambda_{quad}(W)$ as a twisted group algebra defined by the commutation relations $e_{s_\gamma} \cdot w = (-1)^{l(w)} w \cdot e_{s_{w\gamma}}$, where s_γ is a reflection with respect to a root γ and $w \in W$. The coproduct Δ, the antipode S and the counit ε are given by the formulas:

$$\begin{aligned} \Delta(e_{s_\gamma}) &= e_{s_\gamma} \otimes 1 - s_\gamma \otimes e_{s_\gamma}, & \Delta(w) &= w \otimes w, \\ S(e_{s_\gamma}) &= s_\gamma \cdot e_{s_\gamma}, & S(w) &= w^{-1}, \\ \varepsilon(e_{s_\gamma}) &= 0, & \varepsilon(w) &= 1. \end{aligned}$$

The adjoint representation of the Hopf algebra gives an action of $\Lambda_{quad}(W)$ on itself. The element e_{s_γ} acts as a twisted derivation

$$D_\gamma(x) = e_{s_\gamma} x - (-1)^{\deg x} s_\gamma(x) e_{s_\gamma},$$

for a homogeneous element $x \in \Lambda_{quad}(W)$. The twisted derivation D_γ satisfies the twisted Leibniz rule $D_\gamma(xy) = D_\gamma(x)y + (-1)^{\deg x} s_\gamma(x) D_\gamma(y)$.

Remark 4.1. The Hopf algebra considered above coincides with the one obtained as a twisted group algebra over the quadratic lift of the bracket algebra $BE(W,S)$ defined in [4]. (If the root system is simply-laced, the bracket algebra itself is a quadratic algebra.) In particular, it coincides with the fibered Hopf algebra introduced in [3] for the root system of type A.

5. Subalgebra generated by flat connections. In this section, we discuss on the structure of the subalgebra generated by the flat connections θ_α, which are introduced in Section 3. We will treat only classical root systems. Since we use only quadratic relations, we work on the quadratic algebra Λ_{quad}. For simplicity, the symbols (ij), $\overline{(ij)}$ and (i) are used instead of $e_{(ij)}$, $e_{\overline{(ij)}}$ and $e_{(i)}$ respectively. The $U(1)$-connections $\theta_1^X, \dots, \theta_n^X \in \Lambda_{quad}(X)$ ($X = A_{n-1}, B_n, D_n$) are defined as follows:

$$\theta_i^{A_{n-1}} = \sum_{j=1}^n (ij),$$

$$\theta_i^{D_n} = \sum_{j=1}^n ((ij) + \overline{(ij)}),$$

$$\theta_i^{B_n} = \sum_{j=1}^n ((ij) + \overline{(ij)}) + 2(i).$$

One can easily check that the elements $-\theta_i$ define flat connections and satisfy the anticommutativity $\theta_i\theta_j + \theta_j\theta_i = 0$ by direct computations, cf. [4].

Lemma 5.1 (Cyclic relations in $\Lambda_{quad}(A_{n-1})$)**.** *For any distinct* $1 \le a_1, \dots, a_k \le n$,

$$\sum_{i=2}^k (-1)^{k(i-1)} (a_1a_i)(a_1a_{i+1}) \cdots (a_1a_k)(a_1a_2) \cdots (a_1a_i) = 0.$$

Proof. These relations are obtained by applying the composition of twisted derivations $D_{a_{k-1}a_k} D_{a_{k-2}a_{k-1}} \cdots D_{a_2a_3}$ to the relation $(a_1a_2)^2 = 0$. □

Lemma 5.2. *For any distinct* $1 \le a_1, \dots, a_{k+1} \le n$,

$$\begin{aligned} &\left(\prod_{j=2}^k (a_1a_j)\right)(a_1a_2)(a_1a_{k+1}) \\ &+ (-1)^{k+1}(a_1a_{k+1})\left(\prod_{j=2}^k (a_1a_j)\right)(a_1a_2) \\ &+ \left(\prod_{j=2}^{k+1} (a_1a_j)\right)(a_2a_{k+1}) \\ &+ (-1)^{k+1}(a_2a_{k+1})(a_1a_{k+1})\left(\prod_{j=3}^k (a_1a_j)\right)(a_1a_2) \\ &= 0. \end{aligned}$$

Proof. By using the equalities

$$\begin{aligned} &(a_1a_{k+1})(a_2a_{k+1}) + (a_2a_{k+1})(a_1a_2) \\ &\quad + (a_1a_2)(a_1a_{k+1}) = 0, \\ &(a_2a_{k+1})(a_1a_{k+1}) + (a_1a_{k+1})(a_1a_2) \\ &\quad + (a_1a_2)(a_2a_{k+1}) = 0 \end{aligned}$$

and anticommutativity relations, we obtain

$$\begin{aligned} &\left(\prod_{j=2}^k (a_1a_j)\right)(a_1a_2)(a_1a_{k+1}) \\ &+ \left(\prod_{j=2}^{k+1} (a_1a_j)\right)(a_2a_{k+1}) \\ &= -\left(\prod_{j=2}^k (a_1a_j)\right)(a_2a_{k+1})(a_1a_2) \\ &= -(-1)^{k-2}(a_1a_2)(a_2a_{k+1})\left(\prod_{j=3}^k (a_1a_j)\right)(a_1a_2) \\ &= (-1)^k(a_1a_{k+1})\left(\prod_{j=2}^k (a_1a_j)\right)(a_1a_2) \\ &\quad + (-1)^k(a_2a_{k+1})(a_1a_{k+1})\left(\prod_{j=3}^k (a_1a_j)\right)(a_1a_2). \end{aligned}$$

This completes the proof. $\square$

Corollary 5.1. *For any $m \in \mathbf{Z}_{\geq 1}$,*

$$(\theta_1^{A_{n-1}})^{2m} + \cdots + (\theta_n^{A_{n-1}})^{2m} = 0.$$

Proof. The sum $\sum_i \theta_i^{2m}$ is a sum of products of cycles, and the number of odd cycles is even. All even cycles give zero contribution, see Lemma 5.1. According to Lemma 5.2 we can kill all even products of odd cycles. $\square$

Lemma 5.3.

$$\theta_1^{A_{n-1}} \cdots \theta_n^{A_{n-1}} = 0.$$

Lemma 5.4. *For any integer k between 1 and n, we have*

$$\prod_{j=1, j\neq k}^{n} \theta_j^{A_{n-1}} = \sum_{\sigma \in Per(1,\dots,\widehat{k},\dots,n)} (-1)^{l(\sigma)} \prod_{j=1,j\neq k}^{n} (\sigma(j), k),$$

where the sum runs over all permutations σ of the set $(1,\dots,\widehat{k},\dots,n)$ and $l(\sigma)$ denotes the length of permutation σ.

Proof of Lemmas 5.3 *and* 5.4. The proof is by induction on n. We will prove the equations in Lemmas 5.3 and 5.4 for A_{n-1} under the assumption that Lemma 5.3 holds for A_{n-2}. The equation $\theta_1^{A_{n-2}} \cdots \theta_{n-1}^{A_{n-2}} = 0$ means that we have in $\Lambda_{quad}(A_{n-1})$

$$\sum_{i_1,\dots,i_{n-1}} (1, i_1) \cdots (n-1, i_{n-1}) = 0,$$

where $i_1,\dots,i_{n-1}$ run over the letters satisfying $i_l \neq l$ and $1 \leq i_1,\dots,i_{n-1} \leq n-1$. Let M_1 be the sum of the products $\prod_{j=1,j\neq k}^{n}(j, i_j)$ such that none of the letters i_j equal k, and M_2 be the sum of the products such that at least one letter i_j equals k. Then, $\prod_{j=1,j\neq k}^{n} \theta_j^{A_{n-1}} = M_1 + M_2$. The assumption of the induction shows $M_1 = 0$. We can express $\prod_{j=1,j\neq k}^{n}(\sigma(j), k)$ as a sum of terms of form $\pm(1, b_1)\cdots(n, b_n)$ by applying substitution $(a_i b)(a_{i+1} b) \to -(a_{i+1} b)(a_i a_{i+1}) - (a_{i+1} a_i)(a_i b)$ repeatedly when a term $\cdots(a_i b)(a_{i+1} b)\cdots$ with $a_i > a_{i+1}$ appears. This procedure yields the equality

$$\sum_{\sigma \in Per(1,\dots,\widehat{k},\dots,n)} (-1)^{l(\sigma)} \prod_{j=1,j\neq k}^{n} (\sigma(j), k) = M_2.$$

Now we have the equality in Lemma 5.4. Multiply both hand side by $\theta_k^{A_{n-1}}$. Then we have

$$(-1)^{k-1}\theta_1^{A_{n-1}} \cdots \theta_n^{A_{n-1}}$$
$$= \sum_{\sigma \in Per(1,\dots,\widehat{k},\dots,n)} \sum_{l\neq k} (-1)^{l(\sigma)} (k, l) \prod_{j=1,j\neq k}^{n} (\sigma(j), k).$$

Here, we can show that the right hand side is equal to zero from the cyclic relations in Lemma 5.1. $\square$

Lemma 5.5.

$$\sum_{k=1}^{m} (-1)^{(m-1)(k-1)} \prod_{j=k+1}^{m} (k, j) \prod_{j=1}^{k-1} (j, k) = 0.$$

Proof. By induction, one can show

$$D_{a_m a_{m+1}}\left(\prod_{j=1}^{m-1} (a_j a_m)\right)$$
$$= (-1)^m (a_m a_{m+1}) \left(\prod_{j=1}^{m-1} (a_j a_m)\right) + \prod_{j=1}^{m} (a_j a_{m+1})$$

by using the identity

$$\prod_{j=1}^{m-1} (a_j a_m) = (-1)^{n-2} \prod_{j=2}^{m-1} (a_j a_m) \cdot (a_1 a_2)$$
$$- (a_1 a_2)(a_1 a_m) \prod_{j=3}^{m-1} (a_j a_m).$$

Then, the desired identity is obtained by applying $D_{a_{m-1} a_m} \cdots D_{a_3 a_4}$ to the identity $(a_1 a_2)(a_1 a_3) + (a_2 a_3)(a_1 a_2) + (a_1 a_3)(a_2 a_3) = 0$. $\square$

Theorem 5.1. *The connections $\theta_1^{A_{n-1}},\dots,\theta_n^{A_{n-1}}$ satisfy the following relations*:

$$\epsilon_k((\theta_1^{A_{n-1}})^2, \dots, (\theta_n^{A_{n-1}})^2) = 0, \quad 1 \leq k \leq n,$$

where ϵ_k is the k-th elementary symmetric polynomial. Moreover,

$$\theta_1^{A_{n-1}} \cdots \theta_n^{A_{n-1}} = 0,$$
$$\sum_{i=1}^{n} (-1)^{i+1} \theta_1^{A_{n-1}} \cdots \hat{\theta}_i^{A_{n-1}} \cdots \theta_n^{A_{n-1}} = 0.$$

Proof. Indeed, the first series of equalities follow from Corollary 5.1. The second equality has been proved in Lemma 5.3. The last relation follows from Lemmas 5.4 and 5.5. $\square$

Let us remark that

$$\epsilon_{n-1}((\theta_1^{A_{n-1}})^2, \dots, (\theta_n^{A_{n-1}})^2)$$
$$= \left(\sum_{i=1}^{n} (-1)^{i+1} \theta_1^{A_{n-1}} \cdots \hat{\theta}_i^{A_{n-1}} \cdots \theta_n^{A_{n-1}}\right)^2.$$

Proposition 5.1. *The elements* $E_{(ij)} := e_{(ij)} + e_{\overline{(ij)}} \in \Lambda_{quad}(D_n)$ *generate a subalgebra isomorphic to* $\Lambda_{quad}(A_{n-1})$*, where we have the natural identification* $\theta_j^{A_{n-1}} = \theta_j^{D_n}$, $1 \leq j \leq n$.

Proof. We can check the identities $E_{(ij)}^2 = 0$, $E_{(ij)}E_{(kl)} + E_{(kl)}E_{(ij)} = 0$, for $\{i,j\} \cap \{k,l\} = \emptyset$, $E_{(ij)}E_{(jk)} + E_{(jk)}E_{(ki)} + E_{(ki)}E_{(ij)} = 0$. Hence, we can define an algebra homomorphism $\iota : \Lambda_{quad}(A_{n-1}) \to \Lambda_{quad}(D_n)$ by mapping $e_{(ij)}$ to $E_{(ij)}$. We also have an algebra homomorphism $\pi : \Lambda_{quad}(D_n) \to \Lambda_{quad}(A_{n-1})$ obtained by putting $e_{\overline{(ij)}} = 0$. Since $\pi \circ \iota = \mathrm{id}$, the elements $E_{(ij)}$ generate a subalgebra isomorphic to $\Lambda_{quad}(A_{n-1})$. $\square$

Corollary 5.2.

$$\epsilon_k((\theta_1^{D_n})^2, \ldots, (\theta_n^{D_n})^2) = 0, \quad 1 \leq k \leq n.$$

Moreover, $\theta_1^{D_n} \cdots \theta_n^{D_n} = 0$,

$$\sum_{i=1}^{n} (-1)^{i+1} \theta_1^{D_n} \cdots \hat{\theta}_i^{D_n} \cdots \theta_n^{D_n} = 0.$$

Conjecture 5.1. (1) *Let* X *denote either* A_{n-1} *or* D_n*. Relations*

$$\epsilon_k((\theta_1^{X})^2, \ldots, (\theta_n^{X})^2) = 0, \quad 1 \leq k \leq n.$$

$$\theta_1^X \cdots \theta_n^X = 0,$$

$$\sum_{i=1}^{n} (-1)^{i+1} \theta_1^{X} \cdots \hat{\theta}_i^{X} \cdots \theta_n^{X} = 0$$

together with the anticommutativity relations $\theta_i^X \theta_j^X + \theta_j^X \theta_i^X = 0$*, form the complete list of relations among* $\theta_1^X, \ldots, \theta_n^X$ *in the quadratic algebra* $\Lambda_{quad}(X)$.

(2) *For* $X = B_n$*, the relations*

$$\epsilon_k((\theta_1^{X})^2, \ldots, (\theta_n^{X})^2) = 0, \quad 1 \leq k \leq n$$

and the anticommutativity relations form the complete list of relations among $\theta_1^{B_n}, \ldots, \theta_n^{B_n}$ *in the algebra* $\Lambda_{quar}(B_n)$.

We can check that the above relations are valid in the algebra $\Lambda_{quar}(B_n)$ for $n \leq 3$.

Remark 5.1. Let us consider the flag variety Fl_n of type A_{n-1} and the tautological flag on it:

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathcal{O}_{Fl_n}^{\oplus n}.$$

The cohomology ring $H^*(Fl_n, K)$ is isomorphic to the algebra $K[x_1, \ldots, x_n]/(\epsilon_1(x), \ldots, \epsilon_n(x))$, where $x_i = c_1(F_i/F_{i-1})$. The algebra generated by the flat connections $\theta_i^{A_{n-1}}$ can be considered as a super-analogue of the cohomology ring of the flag variety, and our result shows that both algebras have some common relations in even degrees.

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