On conformally flat critical Riemannian metrics for a curvature functional

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Abstract: The normalized L^2-norm of the traceless part of the Ricci curvature defines a Riemannian functional on the space of Riemannian metrics. In this paper, we will consider the critical Riemannian metrics with a flat conformal structure for this functional.

Key words: Critical Riemannian metrics; Riemannian functionals.

1. Introduction. Let M be a closed oriented smooth n-manifold, $\mathcal{M}(M)$ the space of smooth Riemannian metrics on M, and $\mathrm{Diff}(M)$ the diffeomorphism group. We consider a functional $F : \mathcal{M}(M) \to \mathbf{R}$ defined by

$$F(g) = \left(\int_M dv_g\right)^{\frac{4-n}{n}} \int_M \left|\mathrm{Ric}_g - \frac{R_g}{n} g\right|_g^2 dv_g$$
$$= \left(\int_M dv_g\right)^{\frac{4-n}{n}} \int_M |Z_g|_g^2 dv_g,$$

where dv_g, Ric_g, R_g, and $Z_g := \mathrm{Ric}_g - (1/n)R_g g$ denote the volume element, the Ricci curvature, the scalar curvature, and the traceless part of the Ricci curvature of (M, g), respectively. The functional F is Riemannian, that is, for all $g \in \mathcal{M}(M)$, $\varphi \in \mathrm{Diff}(M)$, and any positive constant c, $F(\varphi^* g) = F(cg) = F(g)$.

Obviously, if g is an Einstein metric, then g is a critical point for F with the critical value zero. It is well known that a critical point of the total scalar curvature functional $I : \mathcal{M}(M) \to \mathbf{R}$ defined by

$$I(g) = \left(\int_M dv_g\right)^{\frac{2-n}{n}} \int_M R_g dv_g$$

is exactly an Einstein metric. However, it is difficult to obtain an existence theory for critical points for the total scalar curvature functional I. Since the Riemann curvature is needed to control the convergence or degeneration of Riemannian metrics, the scalar curvature alone being too weak, it is perhaps natural to consider other Riemannian functionals ([1]). Especially in dimension 3, it seems that one can controll the Riemann curvature of a critical Riemannian metric for the functional F. On the other hand, we do not know complete geometric properties of critical Riemannian metrics for the functional F [2].

In this paper, we will prove that a critical Riemannian metric for F must be an Einstein metric under some conditions.

2. Preliminaries. We denote the pointwise inner product and norm with respect to g by $(\cdot,\cdot)_g$ and $|\cdot|_g$ respectively.

Lemma 2.1. *Let $\{g(t)\} \subset \mathcal{M}(M)$ be a one-parameter family of Riemannian metrics with $g(0) = g$ and $(d/dt)g(t)|_{t=0} = h$. Then*

$$\frac{d}{dt}\int_M |Z_{g(t)}|_g^2 dv_{g(t)}\bigg|_{t=0} = \int_M (h, L_g)_g \, dv_g,$$

$$L_g := \bar{\Delta}_g Z_g - 2\,\mathrm{Rm}_g \cdot Z_g + \frac{n-2}{n}\nabla^2 R_g + \frac{n-2}{2n}(\Delta_g R_g)g + \frac{1}{2}|Z_g|_g^2 g,$$

where $\bar{\Delta}_g = \delta_g \nabla = -\nabla^k \nabla_k$ and $\Delta_g = \delta_g d = -\nabla^k \nabla_k$ is the rough Laplacian and the Laplacian, ∇^2 is the Hessian, and $(\mathrm{Rm}_g \cdot Z_g)_{ij} = R_{ikjm} Z_{lp} g^{kl} g^{mp}$.

Proof. Direct calculation by using the known formulae

$$\frac{d}{dt}(\mathrm{Ric}_{g(t)})_{ij}\bigg|_{t=0}$$
$$= \frac{1}{2}(-\nabla^k \nabla_k h_{ij} - 2R_{ikjm}h^{km} + R_{ik}h_j^k + R_{jk}h_i^k + \nabla_i \nabla_k h_j^k + \nabla_j \nabla_k h_i^k - \nabla_i \nabla_j h_k^k),$$

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$$\left.\frac{d}{dt}R_{g(t)}\right|_{t=0} = \Delta_g(\mathrm{tr}_g\, h) + \delta_g\delta_g h - (\mathrm{Ric}_g, h)_g,$$

and

$$\left.\frac{d}{dt}dv_{g(t)}\right|_{t=0} = \frac{1}{2}\,\mathrm{tr}_g\, h dv_g.$$

□

Corollary 2.2. *A Riemannian metric $g \in \mathcal{M}(M)$ is critical for F if and only if g satisfies the following Euler-Lagrange equations*:

$$L^1_g := \bar{\Delta}_g Z_g - 2\left(\mathrm{Rm}_g \cdot Z_g - \frac{|Z_g|^2_g}{n}g\right) + \frac{n-2}{n}\left(\nabla^2 R_g + \frac{\Delta_g R_g}{n}g\right) = 0,$$

$$L^2_g := \frac{(n-2)^2}{n}\Delta_g R_g + (n-4)|Z_g|^2 - c = 0,$$

and $c = (n-4)F(g)\,\mathrm{Vol}(M,g)^{-4/n}$ is the constant, where $\mathrm{Vol}(M,g)$ *is the volume of g.*

Proof. From the Lagrange multiplier argument, we have $L_g = \lambda g$ for some constant λ. Taking trace of this equation, we have $L^2_g = 0$, and so $L^1_g = 0$. Integrating the equation $L^2_g = 0$ over M, we get $c = (n-4)F(g)\,\mathrm{Vol}(M,g)^{-4/n}$. □

Corollary 2.3. *If* $\dim M = 4$ *and* $g \in \mathcal{M}(M)$ *is a critical Riemannian metric for F, then the scalar curvature R_g of g is constant.*

Remark 2.4. The gradient vector $(\nabla F)_g$ of F at $g \in \mathcal{M}(M)$ is in the tangent space $T_g\mathcal{M}(M) \cong \Gamma(S^2T^*M)$, the space of symmetric tensors of type (0,2). The space $T_g\mathcal{M}(M)$ has the trivial pointwise orthogonal splitting with respect to g:

$$T_g\mathcal{M}(M) \cong \Gamma^T_g(S^2T^*M) \oplus C^\infty(M)\cdot g,$$

where $\Gamma^T_g(S^2T^*M)$ is the space of traceless symmetric tensors of type (0,2), and $C^\infty(M)$ is the space of smooth functions on M. Up to constant, L^1_g and L^2_g are orthogonal projection of $(\nabla F)_g$ on these spaces.

For the functional F, we have known the following results.

Theorem 2.5 ([6])**.** *Let M be a closed 3-manifold, and $g \in \mathcal{M}(M)$ a critical Riemannian metirc for F with positive constant scalar curvature. If $|Z_g|^2_g \leq (1/26)R^2_g$, then g is an Einstein metric.*

Theorem 2.6 ([4] cf. [3], [5])**.** *There is a critical Riemannian metric g on the 3-sphere S^3 for F which is of Berger type but not of Einstein.*

Theorem 2.7 ([4])**.** *Let M be a closed 3-manifold, and $g \in \mathcal{M}(M)$ a critical Riemannian metric for F with a flat conformal structure. If the total scalar curvature functional is nonnegative: $I(g) \geq 0$, then g is an Einstein metric.*

3. Critical Riemannian metrics with a flat conformal structure.

Theorem 3.1. *Let $g \in \mathcal{M}(M)$ be a critical Riemannian metric for F with a flat conformal structure. If*

(a) $\dim M = 4$ *and* $I(g) \neq 0$, *or*
(b) $\dim M = 5$ *and* $I(g) \leq 0$, *or*
(c) $\dim M = 2k$ *and R_g is negative constant, or* $\dim M = 2k+1$ *and R_g is nonpositive constant, where $k \geq 3$,*

then g is an Einstein metric.

Lemma 3.2. *Let (M, g) be a conformally flat n-manifold. Then*

$$\Lambda_g := -\bar{\Delta}_g Z_g - \frac{n-2}{2(n-1)}\left(\nabla^2 R_g + \frac{\Delta_g R_g}{n}g\right) - \frac{n}{n-2}\left(Z_g\cdot Z_g - \frac{|Z_g|^2_g}{n}g\right) - \frac{R_g}{n-1}Z_g = 0,$$

where $(Z_g\cdot Z_g)_{ij} = Z_{ik}Z^k_j$.

Proof. Since (M, g) is conformally flat, Cotton-York tensor (Schouten-Weyl tensor) vanishes ([2]), i.e.

$$\nabla_k Z_{ij} - \nabla_j Z_{ik} + \frac{n-2}{2n(n-1)}(\nabla_k R g_{ij} - \nabla_j R g_{ik}) = 0.$$

Differentiating and contracting this equation and from the second Bianchi identity, we have

$$\begin{aligned}
0 &= -\bar{\Delta}Z_{ij} - \nabla^k\nabla_j Z_{ik} + \frac{n-2}{2n(n-1)}(-\Delta R g_{ij} - \nabla_i\nabla_j R)\\
&= -\bar{\Delta}Z_{ij} - \nabla_j\nabla_k Z^k_i - R^k_{lkj}Z^l_i + R^l_{ikj}Z^k_l + \frac{n-2}{2n(n-1)}(-\Delta R g_{ij} - \nabla_i\nabla_j R)\\
&= -\bar{\Delta}Z_{ij} - \frac{n-2}{2n}\nabla_i\nabla_j R - R_{lj}Z^l_i + R^l_{ikj}Z^k_l + \frac{n-2}{2n(n-1)}(-\Delta R g_{ij} - \nabla_i\nabla_j R).
\end{aligned}$$

Using

$$R^l_{ikj} = \frac{1}{n-2}(\delta^l_k Z_{ij} - \delta^l_j Z_{ik} + Z^l_k g_{ij} - Z^l_j g_{ik}) + \frac{R}{n(n-1)}(\delta^l_k g_{ij} - \delta^l_j g_{ik}),$$

we get the desired result. □

Lemma 3.3. *Let $g \in \mathcal{M}(M)$ be a critical Riemannian metric for F with a flat conformal structure, then the Euler-Lagrange equation is the following form.*

$$L_g^1 = \bar{\Delta}_g Z_g + \frac{n-2}{n}\left(\nabla^2 R_g + \frac{\Delta_g R_g}{n} g\right) + \frac{4}{n-2}\left(Z_g \cdot Z_g - \frac{|Z_g|_g^2}{n} g\right) + \frac{2R_g}{n(n-1)} Z_g = 0,$$

$$L_g^2 = \frac{(n-2)^2}{n}\Delta_g R_g + (n-4)|Z_g|_g^2 - c = 0.$$

Proof. If g is conformally flat, then

$$\mathrm{Rm}_g \cdot Z_g = \frac{1}{n-2}(|Z_g|_g^2 g - 2Z_g \cdot Z_g) - \frac{R_g}{n(n-1)} Z_g.$$

□

Proof of Theorem 3.1. Let $\dim M = 4$. Then R_g is constant. Since $L_g^1 = 0$ and $\Lambda_g = 0$, we have $R_g Z_g = 0$. If $I(g) \neq 0$, then $Z_g = 0$, i.e. g is an Einstein metric.

Let $\dim M \geq 5$. Since $L_g^1 = 0$ and $\Lambda_g = 0$, we have

$$(n-4)\bar{\Delta}_g Z_g + \frac{(n-2)(n-3)}{n-1}\left(\nabla^2 R_g + \frac{\Delta_g R_g}{n} g\right) - \frac{2R_g}{n-1} Z_g = 0.$$

Taking inner product of this equation with Z_g and integral by parts, we have

$$\int_M \left((n-4)|\nabla Z_g|_g^2 - \frac{(n-2)^2(n-3)}{2n(n-1)}|dR_g|_g^2 - \frac{2R_g}{n-1}|Z_g|_g^2\right) dv_g = 0.$$

Multipling the equation $L_g^2 = 0$ by R_g, and integrating the result over M, we get

$$\int_M R_g |Z_g|_g^2 dv_g = \frac{1}{n-4}\int_M \left(cR_g - \frac{(n-2)^2}{n}|dR_g|_g^2\right) dv_g.$$

Therefore

$$\int_M \left((n-4)|\nabla Z_g|_g^2 - \frac{(n-2)^2(n^2-7n+8)}{2n(n-1)(n-4)}|dR_g|_g^2\right) dv_g = \frac{2c}{(n-1)(n-4)} I(g) \operatorname{Vol}(M,g)^{(n-2)/n}.$$

If $\dim M = 5$ and $I(g) \leq 0$, then we know that R_g is nonpositive constant and the Ricci curvature is parallel. If $\dim M \geq 6$ and R_g is nonpositive constant, then the Ricci curvature is also parallel.

If $R_g < 0$, then $c = (n-4)F(g)\operatorname{Vol}(M,g)^{-4/n} = 0$, and g is an Einstein metric of the hyperbolic space form.

If $R_g = 0$, then $Z_g \cdot Z_g = (1/n)|Z_g|_g^2 g$. We choose a local frame of orthonormal vector fields adopted to g such that $Z_{ij} = \lambda_i \delta_{ij}$. The values of λ_i are the eigenvalues of the traceless part of the Ricci curvature of g. Simple linear algebraic argument shows that $\lambda_1 + \cdots + \lambda_n = 0$, and $\lambda_1^2 = \cdots = \lambda_n^2$. If $n = \dim M$ is odd, then $\lambda_1 = \cdots = \lambda_n = 0$, that is $Z_g = 0$, and g is an Einstein metric of the Euclidean space form. □

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