

Smooth complete toric threefolds with no nontrivial nef line bundles

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Abstract: We describe all of smooth complete toric threefolds of Picard number 5 with no nontrivial nef line bundles, and show that no such examples exist with Picard number less than 5.

Key words: Projectivity; toric geometry.

1. Introduction. Let X be a complete algebraic variety. A line bundle L on X is said to be nef if $(L \cdot C)$, the degree of L restricted to C , is nonnegative for every curve $C \subset X$. In particular, any ample line bundle on X is nef, as is the trivial line bundle \mathcal{O}_X . Hence complete varieties with no nontrivial nef line bundles are nonprojective. Many constructions of smooth or mildly singular nonprojective varieties use toric geometry; in combinatorial language, they correspond to nonregular triangulations of totally cyclic vector configurations, which are also referred to as nonprojective or noncoherent triangulations. The standard examples of complete nonprojective toric varieties [1], [4], [7, II.2E], [13, pp. 84–85] arise from nonregular triangulations of boundary complexes of polytopes; such varieties admit proper birational morphisms to projective toric varieties, and hence have nontrivial nef line bundles. If Δ is the fan over the faces of a triangulation of the boundary complex of some polytope P , then $X(\Delta)$ is proper and birational over the projective toric variety X_{P° corresponding to the polar polytope P° , and the pullbacks of ample line bundles from X_{P° are nontrivial nef line bundles on $X(\Delta)$.

If a variety has no nontrivial nef line bundles, then it admits no nonconstant morphisms to projective varieties. One example of such a variety, due to Fulton, is the singular toric threefold corresponding to the fan over a cube with one ray displaced, which has no nontrivial line bundles at all [6, pp. 25–26, 72]. A similar example is due to Eikelberg [3, Example 3.5]. The literature also contains a number of examples of Moishezon spaces of Picard number 1 with

no nontrivial nef line bundles. See [1], [9, 5.3.14], [11, 3.3], and [14]. Recently, the first author has given examples of complete, singular toric varieties with arbitrary Picard number that have no nontrivial nef line bundles [5]. Motivated by the observation that all of the standard examples of smooth, complete, nonprojective toric varieties are proper over projective varieties, he conjectured that every smooth, complete toric variety has a nontrivial nef line bundle. The second author found counterexamples to this conjecture, but the examples were relatively complicated, with Picard number 11 and higher [15]. In general, it is known that every smooth complete toric variety with Picard number less than 4 is projective [8].

This note describes the simplest possible examples of smooth complete toric threefolds with no nontrivial nef line bundles, which have Picard number 5. These examples consist of two infinite collections of combinatorially equivalent varieties, plus one exceptional example. The constructions yield similar examples with any Picard number greater than 5, and we show that no such examples exist with Picard number less than 5. This resolves [5, Problem 5.2], which asked for which Picard numbers there exist smooth complete algebraic varieties with no nontrivial nef line bundles, for the case of toric threefolds.

2. Examples. We begin by giving examples of smooth toric threefolds with no nontrivial nef line bundles and Picard number 5.

Example 1. Let Σ be the fan in \mathbf{R}^3 whose rays are generated by

$$\begin{aligned} v_1 &= (1, 0, 0), & v_2 &= (0, 1, 0), & v_3 &= (0, 0, 1) \\ v_4 &= (0, -1, -1), & v_5 &= (-1, 0, -1), & v_6 &= (-2, -1, 0), \end{aligned}$$

and whose maximal cones are

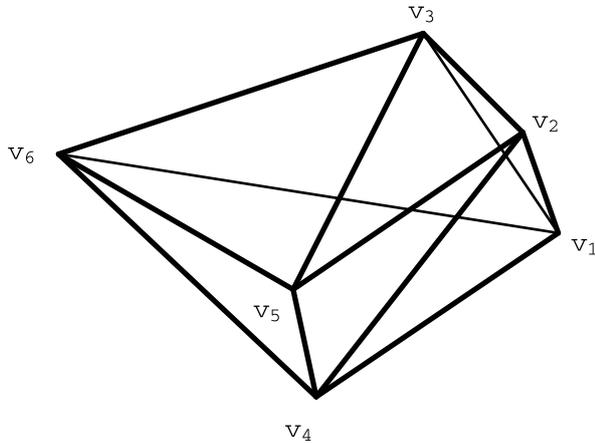
$$\begin{aligned} \langle v_1, v_2, v_3 \rangle, \langle v_1, v_2, v_4 \rangle, \langle v_2, v_4, v_5 \rangle, \langle v_2, v_3, v_5 \rangle, \\ \langle v_3, v_5, v_6 \rangle, \langle v_1, v_3, v_6 \rangle, \langle v_1, v_4, v_6 \rangle, \langle v_4, v_5, v_6 \rangle. \end{aligned}$$

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The nonzero cones of Σ are the cones over the faces of the following nonconvex polyhedron.



Let Δ be the fan obtained from Σ by successive star subdivisions along the rays spanned by $v_7 = (-1, -1, -1)$ and $v_8 = (-2, -1, -1)$.

We claim that $X = X(\Delta)$, the toric threefold corresponding to the fan Δ with respect to the lattice $\mathbf{Z}^3 \subset \mathbf{R}^3$, is smooth and complete and that X has no nontrivial nef line bundles. It is easy to check that $|\Delta| = \mathbf{R}^3$, so X is complete, and that each of the 12 maximal cones of Δ is spanned by a basis for \mathbf{Z}^3 , so X is smooth. It remains to show that X has no nontrivial nef line bundles.

Suppose $D = \sum d_i D_i$ is a nef Cartier divisor, where D_i is the prime T -invariant divisor corresponding to the ray spanned by v_i . Since nef line bundles on toric varieties are globally generated, $\mathcal{O}(D)$ has nonzero global sections. Therefore we may assume that D is effective, i.e. each $d_i \geq 0$. We will show that $D = 0$. Note that $\langle v_1, v_2, v_4 \rangle$ is a cone in Δ , and

$$v_5 = v_2 + v_4 - v_1.$$

It then follows from the convexity of the piecewise linear function Ψ_D associated to D that

$$d_1 + d_5 \geq d_2 + d_4.$$

Similarly,

$$\begin{aligned} d_2 + d_6 &\geq 2d_3 + 2d_5, \\ d_3 + d_4 &\geq 2d_1 + d_6, \end{aligned}$$

Adding the three above inequalities, we have

$$\begin{aligned} d_1 + d_2 + d_3 + d_4 + d_5 + d_6 &\geq \\ 2d_1 + d_2 + 2d_3 + d_4 + 2d_5 + d_6. \end{aligned}$$

Since all of the d_i are nonnegative, it follows that $d_1 = d_3 = d_5 = 0$. Substituting $d_1 = d_5 = 0$ in the first inequality gives $d_2 = d_4 = 0$. Since $\{v_1, \dots, v_5\}$ positively spans \mathbf{R}^3 , we have $P_D = \{0\}$, where P_D is the polytope defined by

$$P_D = \{u \in \mathbf{R}^3 : \langle u, v_i \rangle \geq -d_i \text{ for all } i\}.$$

Since D is nef,

$$-d_i = \min\{\langle u, v_i \rangle : u \in P_D\},$$

for all i . Therefore, $D = 0$.

The above argument shows that any complete fan in \mathbf{R}^3 containing the cones $\langle v_1, v_2, v_4 \rangle$, $\langle v_2, v_3, v_5 \rangle$, and $\langle v_1, v_3, v_6 \rangle$ corresponds to a complete toric variety with no nontrivial nef line bundles. In particular, star subdivisions of the remaining cones in Δ lead to examples of smooth complete toric threefolds with no nontrivial nef line bundles and any Picard number greater than or equal to 5.

Example 1 is the toric variety labeled as (8-12) in [12, p. 79], as can be checked by setting $n = (-1, -1, -1)$, $n' = (1, 0, 0)$, and $n'' = (0, 0, 1)$.

Example 2. Let a be an integer, with $a \neq 0, -1$. Let Δ be the fan whose rays are spanned by

$$\begin{aligned} v_1 &= (1, 0, 0), & v_2 &= (0, 1, 0), & v_3 &= (0, 0, 1), \\ v_4 &= (0, -1, -a), & v_5 &= (0, 0, -1), & v_6 &= (-1, 1, -1), \\ v_7 &= (-1, 0, -1), & v_8 &= (-1, -1, 0), \end{aligned}$$

and whose maximal cones are

$$\begin{aligned} &\langle v_1, v_2, v_3 \rangle, \langle v_1, v_3, v_4 \rangle, \langle v_1, v_4, v_5 \rangle, \langle v_1, v_5, v_6 \rangle, \\ &\langle v_1, v_2, v_6 \rangle, \langle v_2, v_3, v_8 \rangle, \langle v_3, v_4, v_8 \rangle, \langle v_4, v_5, v_8 \rangle, \\ &\langle v_5, v_6, v_7 \rangle, \langle v_5, v_7, v_8 \rangle, \langle v_6, v_7, v_8 \rangle, \langle v_2, v_6, v_8 \rangle. \end{aligned}$$

It is straightforward to check that $X(\Delta)$ is smooth and complete. We claim that $X(\Delta)$ has no nontrivial nef line bundles. Suppose $D = \sum d_i D_i$ is an effective nef divisor on $X(\Delta)$. The convexity of Ψ_D gives the following inequalities:

$$\begin{aligned} d_1 + d_8 &\geq ad_3 + d_4, \\ 2d_4 + d_6 &\geq (2a + 1)d_5 + d_8, \\ d_2 + d_5 &\geq d_1 + d_6, \\ d_3 + d_6 &\geq 2d_2 + d_8, \\ d_2 + d_4 &\geq -ad_3, \\ d_1 + d_8 &\geq d_4 - ad_5. \end{aligned}$$

We claim that $d_3 = d_5 = 0$. When $a > 0$, the claim follows from the first four inequalities, since

$$\begin{aligned} 2(d_1 + d_8) + 2d_4 + d_6 + 2(d_2 + d_5) + d_3 + d_6 &\geq \\ 2(ad_3 + d_4) + (2a + 1)d_5 + d_8 + & \\ 2(d_1 + d_6) + 2d_2 + d_8. & \end{aligned}$$

Similarly, the claim follows from the last four inequalities when $a < -1$. Substituting 0 for d_3 and d_5 in the middle two inequalities gives $d_1 = d_2 = d_8 = 0$. Since $\{v_1, v_2, v_3, v_5, v_8\}$ positively spans \mathbf{R}^3 , it follows that $P_D = \{0\}$, and hence $D = 0$. So $X(\Delta)$ has no nontrivial nef line bundles.

The varieties in Example 2 are labeled (8-5') in [12, p. 78], as can be checked by setting $n = (1, 0, 0)$, $n' = (0, 1, 0)$, and $n'' = (0, 0, 1)$.

Example 3. Let a and b be nonzero integers, with $(a, b) \neq (\pm 1, \pm 1)$. Let Δ be the fan whose rays are spanned by

$$\begin{aligned} v_1 = (-1, b, 0), \quad v_2 = (0, -1, 0), \quad v_3 = (1, -1, 0), \\ v_4 = (-1, 0, -1), \quad u_5 = (0, 0, -1), \quad v_6 = (0, 1, 0), \\ v_7 = (0, 0, 1), \quad v_8 = (1, 0, a), \end{aligned}$$

and whose maximal cones are

$$\begin{aligned} \langle v_1, v_2, v_4 \rangle, \langle v_2, v_3, v_4 \rangle, \langle v_3, v_4, v_5 \rangle, \langle v_4, v_5, v_6 \rangle, \\ \langle v_1, v_4, v_6 \rangle, \langle v_1, v_6, v_7 \rangle, \langle v_1, v_2, v_7 \rangle, \langle v_2, v_3, v_7 \rangle, \\ \langle v_3, v_5, v_8 \rangle, \langle v_5, v_6, v_8 \rangle, \langle v_3, v_7, v_8 \rangle, \langle v_6, v_7, v_8 \rangle. \end{aligned}$$

It is straightforward to check that $X(\Delta)$ is smooth and complete. We claim that $X(\Delta)$ has no nontrivial nef line bundles. Suppose $D = \sum d_i D_i$ is an effective nef divisor on $X(\Delta)$. We now consider the case where a and b are both positive, and show that $D = 0$. The convexity of Ψ_D gives the following inequalities:

$$\begin{aligned} d_3 + d_6 &\geq ad_5 + d_8, \\ d_2 + d_8 &\geq d_3 + ad_7, \\ d_1 + d_5 &\geq bd_6 + d_4, \\ d_4 + d_7 &\geq d_1 + bd_2. \end{aligned}$$

Since $(a, b) \neq (1, 1)$, either $a > 1$ or $b > 1$. We claim that $d_2 = d_5 = d_6 = d_7 = 0$. If $a > 1$, then adding the above inequalities gives $d_5 = d_7 = 0$, and substituting 0 for d_5 and d_7 in the last two inequalities gives $d_2 = d_6 = 0$. If $b > 1$, then adding the above inequalities gives $d_2 = d_6 = 0$, and substituting 0 for d_2 and d_6 in the first two inequalities gives $d_5 = d_7 = 0$. This proves the claim. Now, convexity of Ψ_D gives

$$d_2 + d_5 \geq d_3 + d_4,$$

so $d_3 = d_4 = 0$. Since $\{v_2, \dots, v_7\}$ positively spans \mathbf{R}^3 , it follows that $P_D = \{0\}$, and hence $D = 0$.

The cases where a or b is negative, or both, are proven similarly, using the inequalities above, together with the following inequalities, which also come from the convexity of Ψ_D :

$$\begin{aligned} d_3 + d_6 &\geq -ad_7 + d_8, \\ d_4 + d_8 &\geq (1 - a)d_5, \\ d_4 + d_7 &\geq d_1 - bd_6, \\ d_1 + d_3 &\geq (1 - b)d_2. \end{aligned}$$

The varieties in Example 3 are labeled (8-13') in [12, p. 79], as can be checked by setting $n = (1, -1, 0)$, $n' = (0, 1, 0)$, and $n'' = (0, 0, 1)$.

3. Classification. Using the classification of minimal smooth toric threefolds with Picard number at most 5, due to Miyake and Oda [12, §9], and Nagaya [10], we show that Examples 1, 2, and 3 give all such varieties that have no nontrivial nef line bundles. Recall that a smooth toric variety is minimal if it is not the blow up of a smooth toric variety along a smooth torus invariant center.

Theorem. *Every smooth complete toric threefold with no nontrivial nef line bundle and Picard number at most 5 is isomorphic to one described in Examples 1, 2, and 3, above.*

It will suffice to show that there are no minimal smooth complete toric threefolds with no nontrivial nef line bundles and Picard number less than 5, and that every minimal smooth complete nonprojective toric threefold with Picard number 5 that is not listed in Examples 1, 2, and 3, admits a nonconstant morphism to a projective variety.

Lemma. *There are no minimal smooth complete toric threefolds with no nontrivial nef line bundles and Picard number less than 5.*

Proof. By the classification theorem of Miyake, Oda, and Nagaya, every minimal smooth complete toric threefold with Picard number less than 5 is isomorphic to either \mathbf{P}^3 , a \mathbf{P}^2 -bundle over \mathbf{P}^1 , a \mathbf{P}^1 -bundle over a projective toric surface, or one of the following

- (a) The projective toric variety $X(\Delta)$, where Δ is the fan in \mathbf{R}^3 whose rays are generated by

$$\begin{aligned} v_1 = (1, 0, 0), \quad v_2 = (0, 1, 0), \\ v_3 = (0, 0, 1), \quad v_4 = (0, -1, a), \\ v_5 = (0, 0, -1), \quad v_6 = (0, 1, -1), \\ v_7 = (-1, 2, -1), \end{aligned}$$

for some integer a , and whose maximal cones are

$$\begin{aligned} &\langle v_1, v_2, v_3 \rangle, \langle v_1, v_3, v_4 \rangle, \langle v_1, v_4, v_5 \rangle, \langle v_1, v_5, v_6 \rangle, \\ &\langle v_1, v_6, v_7 \rangle, \langle v_1, v_2, v_7 \rangle, \langle v_2, v_3, v_7 \rangle, \langle v_3, v_4, v_7 \rangle, \\ &\langle v_4, v_5, v_7 \rangle, \langle v_5, v_6, v_7 \rangle. \end{aligned}$$

- (b) The toric variety X associated to the fan in \mathbf{R}^3 whose rays are generated by

$$\begin{aligned} v_1 &= (-1, 0, 0), v_2 = (0, -1, 0), v_3 = (0, 0, -1), \\ v_4 &= (1, 0, 1), v_5 = (0, 1, 1), \\ v_6 &= (1, 1, 1), v_7 = (1, 1, 0), \end{aligned}$$

and whose maximal cones are

$$\begin{aligned} &\langle v_1, v_2, v_3 \rangle, \langle v_2, v_3, v_4 \rangle, \langle v_2, v_4, v_5 \rangle, \langle v_1, v_2, v_5 \rangle, \\ &\langle v_4, v_5, v_6 \rangle, \langle v_1, v_3, v_7 \rangle, \langle v_3, v_4, v_7 \rangle, \langle v_4, v_6, v_7 \rangle, \\ &\langle v_5, v_6, v_7 \rangle, \langle v_1, v_5, v_7 \rangle. \end{aligned}$$

In this case, X is not projective [12, Proposition 9.4], but the anticanonical bundle $-K_X$ is nef.

Therefore, up to isomorphism, there is only one non-projective smooth complete toric threefold with Picard number less than 5, and it has a nontrivial nef line bundle. \square

Proof of Theorem. It remains to show that every minimal smooth complete nonprojective toric threefold of Picard number 5 that is not isomorphic to one in Examples 1, 2, or 3 has a nontrivial nef line bundle. By the classification theorem of Miyake and Oda, and Nagaya, and by the second Remark in [12, p. 80], any such variety is isomorphic to one of the varieties listed below. In each case, it will suffice to either exhibit a specific nontrivial nef line bundle, or to give a nonconstant morphism to a projective variety. Note that the varieties $X(\Delta)$ as in Examples 2 and 3, but with $a = 0$ or $b = 0$, are not minimal. The labels such as (8-5') are the labels given to the varieties in [12, pp. 78–79]. Different labels with the same numbers, such as (8-5') and (8-5''), indicate that the corresponding varieties have combinatorially equivalent fans.

- (a) (8-5') The toric variety $X = X(\Delta)$ as Example 2, with $a = -1$. In this case, $-K_X - D_4 - D_8$ is nef.
 (b) (8-5'') The toric variety $X(\Delta)$ associated to the fan in \mathbf{R}^3 whose rays are generated by

$$\begin{aligned} v_1 &= (0, 1, 0), & v_2 &= (0, -1, -1), \\ v_3 &= (1, 0, 0), & v_4 &= (0, 0, 1), \\ v_5 &= (-1, 0, -1), & v_6 &= (-1, -2, -2), \\ v_7 &= (-1, -1, -1), & v_8 &= (-1, -1, 0), \end{aligned}$$

and whose maximal cones are

$$\begin{aligned} &\langle v_1, v_2, v_3 \rangle, \langle v_1, v_3, v_4 \rangle, \langle v_1, v_4, v_5 \rangle, \langle v_1, v_5, v_6 \rangle, \\ &\langle v_1, v_2, v_6 \rangle, \langle v_2, v_3, v_8 \rangle, \langle v_3, v_4, v_8 \rangle, \langle v_4, v_5, v_8 \rangle, \\ &\langle v_5, v_6, v_7 \rangle, \langle v_5, v_7, v_8 \rangle, \langle v_6, v_7, v_8 \rangle, \langle v_2, v_6, v_8 \rangle. \end{aligned}$$

In this case, Δ is a refinement of the complete fan whose set of rays is $\{v_1, v_3, v_4, v_7\}$. Therefore $X(\Delta)$ admits a proper birational morphism to \mathbf{P}^3 .

- (c) (8-8) The toric variety $X(\Delta)$ associated to the fan in \mathbf{R}^3 whose rays are generated by

$$\begin{aligned} v_1 &= (0, 0, 1), & v_2 &= (1, 0, 0), \\ v_3 &= (0, -1, -1), & v_4 &= (-1, -2, -1), \\ v_5 &= (0, 1, 0), & v_6 &= (0, 0, -1), \\ v_7 &= (-1, -2, -2), & v_8 &= (-1, -1, -2), \end{aligned}$$

and whose maximal cones are

$$\begin{aligned} &\langle v_1, v_2, v_3 \rangle, \langle v_1, v_3, v_4 \rangle, \langle v_1, v_4, v_5 \rangle, \langle v_1, v_2, v_5 \rangle, \\ &\langle v_2, v_5, v_6 \rangle, \langle v_3, v_4, v_7 \rangle, \langle v_2, v_3, v_8 \rangle, \langle v_3, v_7, v_8 \rangle, \\ &\langle v_4, v_7, v_8 \rangle, \langle v_4, v_5, v_8 \rangle, \langle v_5, v_6, v_8 \rangle, \langle v_2, v_6, v_8 \rangle. \end{aligned}$$

In this case, Δ is a subdivision of the complete fan whose rays are v_1, v_2, v_5, v_7 . Therefore $X(\Delta)$ admits a proper birational morphism to $\mathbf{P}(1, 1, 2, 2)$.

- (d) (8-11) The toric variety $X(\Delta)$ associated to the fan in \mathbf{R}^3 whose rays are generated by

$$\begin{aligned} v_1 &= (1, 0, 0), & v_2 &= (0, -1, 0), \\ v_3 &= (0, 0, 1), & v_4 &= (1, 1, a), \\ v_5 &= (0, 0, -1), & v_6 &= (0, -1, -1), \\ v_7 &= (-1, -2, -1), & v_8 &= (0, 1, b), \end{aligned}$$

for some integers a and b , and whose maximal cones are

$$\begin{aligned} &\langle v_1, v_2, v_3 \rangle, \langle v_1, v_3, v_4 \rangle, \langle v_1, v_4, v_5 \rangle, \langle v_1, v_5, v_6 \rangle, \\ &\langle v_1, v_6, v_7 \rangle, \langle v_1, v_2, v_7 \rangle, \langle v_2, v_3, v_7 \rangle, \langle v_5, v_6, v_7 \rangle, \\ &\langle v_3, v_4, v_8 \rangle, \langle v_4, v_5, v_8 \rangle, \langle v_5, v_7, v_8 \rangle, \langle v_3, v_7, v_8 \rangle. \end{aligned}$$

In this case, the projection $(x, y, z) \mapsto (x, y)$ gives a map from Δ to the complete fan in \mathbf{R}^2 whose rays are generated by $(1, 0), (0, 1), (1, 1)$, and $(-1, -2)$. Therefore $X(\Delta)$ admits a proper surjective morphism to the blow up of $\mathbf{P}(1, 1, 2)$ at a point.

- (e) (8-13') The toric variety $X = X(\Delta)$ as in Example 3, with $(a, b) = (\pm 1, \pm 1)$. In this case, $-K_X$ is nef.
- (f) (8-13'') The toric variety $X(\Delta)$ associated to the fan in \mathbf{R}^3 whose rays are generated by

$$\begin{aligned} v_1 &= (1, 1, b), & v_2 &= (1, 0, 0), \\ v_3 &= (0, -1, 0), & v_4 &= (0, 0, -1), \\ v_5 &= (-1, a, d), & v_6 &= (0, 1, 0), \\ v_7 &= (0, 0, 1), & v_8 &= (-1, c, d + 1), \end{aligned}$$

for some integers a, b, c , and d , and whose maximal cones are

$$\begin{aligned} &\langle v_1, v_2, v_4 \rangle, \langle v_2, v_3, v_4 \rangle, \langle v_3, v_4, v_5 \rangle, \langle v_4, v_5, v_6 \rangle, \\ &\langle v_1, v_4, v_6 \rangle, \langle v_1, v_6, v_7 \rangle, \langle v_1, v_2, v_7 \rangle, \langle v_2, v_3, v_7 \rangle, \\ &\langle v_3, v_5, v_8 \rangle, \langle v_5, v_6, v_8 \rangle, \langle v_3, v_7, v_8 \rangle, \langle v_6, v_7, v_8 \rangle. \end{aligned}$$

In this case, the projection $(x, y, z) \mapsto x$ gives a map from Δ to the complete fan in \mathbf{R} . Therefore $X(\Delta)$ admits a proper surjective morphism to \mathbf{P}^1 .

- (g) (8-14') The toric variety $X(\Delta)$ associated to the fan in \mathbf{R}^3 whose rays are generated by

$$\begin{aligned} v_1 &= (0, 0, -1), & v_2 &= (1, 0, 0), \\ v_3 &= (0, 1, 0), & v_4 &= (-1, -1, a), \\ v_5 &= (-1, -1, a + 1), & v_6 &= (1, 0, 1), \\ v_7 &= (0, 0, 1), & v_8 &= (0, 1, 1), \end{aligned}$$

for some integer a , and whose maximal cones are

$$\begin{aligned} &\langle v_1, v_2, v_3 \rangle, \langle v_1, v_3, v_4 \rangle, \langle v_1, v_2, v_4 \rangle, \langle v_3, v_4, v_5 \rangle, \\ &\langle v_4, v_5, v_6 \rangle, \langle v_2, v_4, v_6 \rangle, \langle v_5, v_6, v_7 \rangle, \langle v_2, v_3, v_8 \rangle, \\ &\langle v_3, v_5, v_8 \rangle, \langle v_5, v_7, v_8 \rangle, \langle v_6, v_7, v_8 \rangle, \langle v_2, v_6, v_8 \rangle. \end{aligned}$$

In this case, the projection $(x, y, z) \mapsto (x, y)$ gives a map from Δ to the complete fan in \mathbf{R}^2 whose rays are generated by $(1, 0)$, $(0, 1)$, and $(-1, -1)$. Therefore $X(\Delta)$ admits a proper surjective morphism to \mathbf{P}^2 .

- (h) (8-14'') The toric variety $X(\Delta)$ associated to the fan in \mathbf{R}^3 whose rays are generated by

$$\begin{aligned} v_1 &= (-1, a, b), & v_2 &= (0, 1, 0), \\ v_3 &= (0, -1, -1), & v_4 &= (0, 0, 1), \\ v_5 &= (1, 0, 1), & v_6 &= (1, 1, 0), \\ v_7 &= (1, 0, 0), & v_8 &= (1, -1, -1), \end{aligned}$$

for some integers a and b , and whose maximal cones are

$$\begin{aligned} &\langle v_1, v_2, v_3 \rangle, \langle v_1, v_3, v_4 \rangle, \langle v_1, v_2, v_4 \rangle, \langle v_3, v_4, v_5 \rangle, \\ &\langle v_4, v_5, v_6 \rangle, \langle v_2, v_4, v_6 \rangle, \langle v_5, v_6, v_7 \rangle, \langle v_2, v_3, v_8 \rangle, \end{aligned}$$

$$\langle v_3, v_5, v_8 \rangle, \langle v_5, v_7, v_8 \rangle, \langle v_6, v_7, v_8 \rangle, \langle v_2, v_6, v_8 \rangle.$$

In this case, the projection $(x, y, z) \mapsto x$ gives a map from Δ to the complete fan in \mathbf{R} . Therefore $X(\Delta)$ admits a proper surjective morphism to \mathbf{P}^1 . □

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References

- [1] L. Bonavero, Sur des variétés de Moishezon dont le groupe de Picard est de rang un, *Bull. Soc. Math. France* **124** (1996), no. 3, 503–521.
- [2] L. Bonavero, Sur des variétés toriques non projectives, *Bull. Soc. Math. France* **128** (2000), no. 3, 407–431.
- [3] M. Eikelberg, The Picard group of a compact toric variety, *Results Math.* **22** (1992), no. 1-2, 509–527.
- [4] G. Ewald, Spherical complexes and nonprojective toric varieties, *Discrete Comput. Geom.* **1** (1986), no. 2, 115–122.
- [5] O. Fujino, On the Kleiman-Mori cone, *Proc. Japan Acad.*, **81A** (2005), no. 5, 80–84.
- [6] W. Fulton, *Introduction to toric varieties*, *Ann. of Math. Stud.*, 131, Princeton Univ. Press, Princeton, NJ, 1993.
- [7] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings*. I, *Lecture Notes in Math.*, 339, Springer, Berlin, 1973.
- [8] P. Kleinschmidt and B. Sturmfels, Smooth toric varieties with small Picard number are projective, *Topology* **30** (1991), no. 2, 289–299.
- [9] J. Kollár, Flips, flops, minimal models, etc., in *Surveys in differential geometry (Cambridge, MA, 1990)*, 113–199, Lehigh Univ., Bethlehem, PA.
- [10] O. Nagaya, Classification of 3-dimensional complete non-singular torus embeddings, Master’s thesis, Nagoya Univ. 1976.
- [11] I. Nakamura, Moishezon threefolds homeomorphic to \mathbf{P}^3 , *J. Math. Soc. Japan* **39** (1987), no. 3, 521–535.
- [12] T. Oda, *Torus embeddings and applications*, *Tata Inst. Fund. Res.*, Bombay, 1978.
- [13] T. Oda, *Convex bodies and algebraic geometry*, Translated from the Japanese, Springer, Berlin, 1988.

- [14] K. Oguiso, Two remarks on Calabi-Yau Moishezon threefolds, *J. Reine Angew. Math.* **452** (1994), 153–161.
- [15] S. Payne, A smooth, complete threefold with no nontrivial nef line bundles, (2005). (Preprint). [math.AG/0501204](#).