Riemannian submersions, minimal immersions and cohomology class

By Bang-Yen Chen

Department of Mathematics, Michigan State University, East Lansing, Michigan 48864-1027, U.S.A.

(Communicated by Shigefumi Mori, M. J. A., Dec. 12, 2005)

Abstract: We prove a simple optimal relationship between Riemannian submersions and minimal immersions; namely, if a Riemannian manifold admits a non-trivial Riemannian submersion with totally geodesic fibers, then it cannot be isometrically immersed in any Riemannian manifold of non-positive sectional curvature as a minimal manifold. Some related results are also presented. In the last section, we introduce a cohomology class for Riemannian submersions and provide an application.

Key words: Riemannian submersion; minimal immersion; cohomology class; totally geodesic fibers.

1. Introduction. Let M and B be Riemannian manifolds with $n = \dim M > \dim B = b > 0$. A *Riemannian submersion* $\pi\colon M \to B$ is a mapping of M onto B satisfying the following two axioms:

(S1) *π has maximal rank*;

(S2) *the differential π_* preserves lengths of horizontal vectors.*

The mappings between Riemannian manifolds satisfying these two axioms were studied by T. Nagano in [10] in terms of fibred Riemannian manifolds. In particular, he derived the fundamental equations analogous to Weingarten's formulas for Riemannian submanifolds. B. O'Neill further studied such mappings in [11] and called them Riemannian submersions.

For each $p \in B$, $\pi^{-1}(p)$ is an $(n-b)$-dimensional submanifold of M. The submanifolds $\pi^{-1}(p)$, $p \in B$, are called fibers. A vector field on M is called vertical if it is always tangent to fibers; and horizontal if it is always orthogonal to fibers. We use corresponding terminology for individual tangent vectors as well. A vector field on M is called *basic* if X is horizontal and π-related to a vector field X_* on B.

Let $\mathcal{H}$ and $\mathcal{V}$ denote the projections of tangent spaces of M onto the subspaces of horizontal and vertical vectors, respectively. We use the same letters to denote the horizontal and vertical distributions.

The simplest type of Riemannian submersions is the projection of a Riemannian product manifold on one of its factors. For such Riemannian submersions, both horizontal and vertical distributions are *totally geodesic distributions*, i.e., both distributions are completely integrable and their leaves are totally geodesic submanifolds.

In this article, a Riemannian manifold M is said to admit a *non-trivial Riemannian submersion* if there exists a Riemannian submersion $\pi\colon M \to B$ from M onto another Riemannian manifold B such that the horizontal and vertical distributions of the submersion are not both totally geodesic distribution.

Clearly, if a Riemannian submersion has totally geodesic fibers, the submersion is non-trivial if and only if the horizontal distribution $\mathcal{H}$ is not a totally geodesic distribution.

By applying an idea from [3, 4] we prove the following sharp relationship between Riemannian submersions and minimal immersions.

Theorem 1. *If a Riemannian manifold admits a non-trivial Riemannian submersion with totally geodesic fibers, then it cannot be isometrically immersed in any Riemannian manifold of non-positive sectional curvature as a minimal submanifold.*

If $\phi_F\colon F \to \mathbf{E}^{m_1}$ and $\phi_B\colon B \to \mathbf{E}^{m_2}$ are minimal isometric immersions of Riemannian manifolds F and B into Euclidean spaces, then the product immersion of ϕ_F and ϕ_B is the immersion:

$$(1.1) \qquad (\phi_F, \phi_B) : F \times B \to \mathbf{E}^{m_1} \oplus \mathbf{E}^{m_2}$$

which carries $(q, p) \in F \times B$ to $(\phi_F(q), \phi_B(b))$. The

2000 Mathematics Subject Classification. Primary 53C40; Secondary 53C12.

Dedicated to Professor Tadashi Nagano on the occasion of his 75th birthday.

product immersion (ϕ_F, ϕ_B) is also a minimal isometric immersion.

Theorem 2. *Let $\pi\colon M \to B$ be a Riemannian submersion with totally geodesic fibers. If M admits a minimal isometric immersion ϕ into a Euclidean space, then locally M is the Riemannian product of a fiber F and the base manifold B and ϕ is the product immersion (ϕ_F, ϕ_B) of some minimal isometric immersions $\phi_F\colon F \to \mathbf{E}^{m_1}$ and $\phi_B\colon B \to \mathbf{E}^{m_2}$ into some Euclidean spaces.*

2. Proof of Theorem 1. Let M be an n-dimensional Riemannian manifold. Denote by R, K and τ the Riemann curvature tensor, the sectional curvature function and the scalar curvature of M, respectively.

Given an orthonormal basis $e_1, \dots, e_n$ of the tangent space T_pM, $p \in M$, the scalar curvature τ of M at p is defined to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{2.1}$$

Assume that M admits a Riemannian immersion: $\pi\colon M \to B$ with $\dim M > \dim B > 0$. Then there exists a $(1,2)$-tensor A, called the *integrability tensor*, on M defined by

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}\mathcal{E}}(\mathcal{H}F) + \mathcal{H}\nabla_{\mathcal{H}\mathcal{E}}(\mathcal{V}F),$$

for vector fields E, F tangent to M. In particular, for any horizontal vector field X and any vertical vector field V, we have

$$A_X V = \mathcal{H}\nabla_X V. \tag{2.2}$$

Now, assume that the Riemannian submersion $\pi\colon M \to B$ has totally geodesic fibers. Then the sectional curvature $K(x \wedge v)$ on M associated with the plane section spanned by a unit horizontal vector x and a unit vertical vector v is given by (cf. [11, p. 465])

$$K(x \wedge v) = \|A_x v\|^2. \tag{2.3}$$

Let us assume that M admits an isometric immersion $\phi\colon M \to \tilde{M}^m$ into a Riemannian m-manifold $\tilde{M}^m$. Denote by $\tilde{R}$ and $\tilde{K}$ the Riemann curvature tensor and the sectional curvature function of $\tilde{M}$, respectively.

It follows from the equation of Gauss that the scalar curvature τ and the squared mean curvature H^2 of M in $\tilde{M}^m$ satisfy (see, for instance, [1])

$$2\tau(p) = n^2H^2 - \|h\|^2 + 2\tilde{\tau}(T_pM), \tag{2.4}$$

where $\|h\|^2$ denotes the squared norm of the second fundamental form h of M in $\tilde{M}^m$ and $\tilde{\tau}(T_pM)$ is defined as

$$\tilde{\tau}(T_pM) = \sum_{1\le i<j\le n} \tilde{K}(e_i, e_j).$$

Let us put

$$\delta = 2\tau - \frac{n^2}{2}H^2 - 2\tilde{\tau}(T_pM). \tag{2.5}$$

Then (2.4) becomes

$$n^2H^2 = 2\delta + 2\|h\|^2. \tag{2.6}$$

If we choose a local orthonormal frame:

$$e_1, \dots, e_b, e_{b+1}, \dots, e_n, e_{n+1}, \dots, e_m$$

such that $e_1, \dots, e_b$ are horizontal vector fields, $e_{b+1}, \dots, e_n$ are vertical vector fields of M, and that e_{n+1} is a unit normal vector field parallel to the mean curvature vector field of M, then (2.6) becomes

$$\left(\sum_{i=1}^n h_{ii}^{n+1}\right)^2 = 2\left[\delta + \sum_{i=1}^n \left(h_{ii}^{n+1}\right)^2 + \sum_{i\ne j}\left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^m \sum_{i,j=1}^n \left(h_{ij}^r\right)^2\right], \tag{2.7}$$

where $h_{ij}^r = \langle h(e_i, e_j), e_r\rangle$, $n+1 \le r \le m$; $1 \le i, j \le n$, are the coefficients of the second fundamental form and $\langle\ ,\ \rangle$ is the inner product on $\tilde{M}^m$.

Equation (2.7) is equivalent to

$$\begin{aligned}(\bar{a}_1 + \bar{a}_2 + \bar{a}_3)^2 &= 2\Bigg[\delta + \bar{a}_1^2 + \bar{a}_2^2 + \bar{a}_3^2 + 2\sum_{1\le i<j\le n}\left(h_{ij}^{n+1}\right)^2 \\ &\quad + \sum_{r=n+2}^m \sum_{i,j=1}^n \left(h_{ij}^r\right)^2 - 2\sum_{2\le j<k\le b} h_{jj}^{n+1}h_{kk}^{n+1} \\ &\quad - 2\sum_{b+1\le s<t\le n} h_{ss}^{n+1}h_{tt}^{n+1}\Bigg],\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}\bar{a}_1 &= h_{11}^{n+1}, \\ \bar{a}_2 &= h_{22}^{n+1} + \dots + h_{bb}^{n+1}, \\ \bar{a}_3 &= h_{b+1b+1}^{n+1} + \dots + h_{nn}^{n+1}.\end{aligned} \tag{2.9}$$

By applying Lemma 3.1 of [2] to (2.8) we obtain

$$\sum_{1\le j<k\le b} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{b+1\le s<t\le n} h_{ss}^{n+1}h_{tt}^{n+1} \ge \frac{\delta}{2} + \sum_{1\le s<t\le n}\left(h_{st}^{n+1}\right)^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{s,t=1}^{n}(h_{st}^{r})^2, \tag{2.10}$$

with equality holding if and only if we have

$$\sum_{i=1}^{b} h_{ii}^{n+1} = \sum_{s=b+1}^{n} h_{ss}^{n+1}. \tag{2.11}$$

From the equation of Gauss and (2.3), we obtain

$$\begin{aligned}\breve{A}_\pi &= \tau - \sum_{1\le j<k\le b} K(e_j\wedge e_k) - \sum_{b+1\le s<t\le n} K(e_s\wedge e_t)\\ &= \tau - \sum_{1\le i<j\le b}\tilde{K}(e_i\wedge e_j) - \sum_{r=n+1}^{m}\sum_{1\le j<k\le b}\left(h_{jj}^r h_{kk}^r - \left(h_{jk}^r\right)^2\right)\\ &\quad - \sum_{b+1\le s<t\le n}\tilde{K}(e_s\wedge e_t) - \sum_{r=n+1}^{m}\sum_{b+1\le s<t<n}\left(h_{ss}^r h_{tt}^r - (h_{st}^r)^2\right),\end{aligned} \tag{2.12}$$

where $\breve{A}_\pi$ is the *submersion invariant* defined by

$$\breve{A}_\pi = \sum_{i=1}^{b}\sum_{s=b+1}^{n}\|A_{e_i}e_s\|^2. \tag{2.13}$$

Therefore, after applying (2.5), (2.10) and (2.12), we find

(2.14)

$$\begin{aligned}\breve{A}_\pi &\le \tau - \tilde{\tau}(T_pM) - \frac{\delta}{2} + \sum_{i=1}^{b}\sum_{s=b+1}^{n}\tilde{K}(e_i\wedge e_s) - \sum_{j=1}^{n}\sum_{t=b+1}^{n}\left(h_{jt}^{n+1}\right)^2\\ &\quad + \sum_{r=n+2}^{m}\left\{\sum_{1\le j<k\le b}\left((h_{jk}^r)^2 - h_{jj}^r h_{kk}^r\right) + \sum_{b+1\le s<t<n}\left((h_{st}^r)^2 - h_{ss}^r h_{tt}^r\right) - \frac{1}{2}\sum_{\alpha,\beta=1}^{n}(h_{\alpha\beta}^r)^2\right\}\\ &= \tau - \tilde{\tau}(T_pM) + \sum_{i=1}^{b}\sum_{s=b+1}^{n}\tilde{K}(e_i\wedge e_s) - \frac{\delta}{2} - \sum_{r=n+1}^{m}\sum_{j=1}^{b}\sum_{t=b+1}^{n}(h_{jt}^r)^2\\ &\quad - \frac{1}{2}\sum_{r=n+2}^{m}\left\{\left(\sum_{j=1}^{b}h_{jj}^r\right)^2 + \left(\sum_{t=b+1}^{n}h_{tt}^r\right)^2\right\}\\ &\le \tau - \tilde{\tau}(T_pM) + \sum_{i=1}^{b}\sum_{s=b+1}^{n}\tilde{K}(e_i\wedge e_s) - \frac{\delta}{2}\\ &= \frac{n^2}{4}H^2 + \sum_{i=1}^{b}\sum_{s=b+1}^{n}\tilde{K}(e_i\wedge e_s).\end{aligned}$$

Hence we obtain

$$\breve{A}_\pi \le \frac{n^2}{4}H^2 + b(n-b)\max\tilde{K}, \tag{2.15}$$

where $\max\tilde{K}(p)$ denotes the maximum value of the sectional curvature function of $\tilde{M}^m$ restricted to plane sections in T_pM.

Now, suppose that the isometric immersion ϕ of M in $\tilde{M}^m$ is minimal and $\tilde{M}^m$ has non-positive sectional curvature, then from (2.13) and (2.15) we obtain that $A_XV = 0$ for any horizontal vector field X and any vertical vector field V in M. Hence, we know from (2.2) that ∇_XV is always vertical. Therefore, for any horizontal vector fields X, Y, the covariant derivative ∇_XY is always horizontal. From this we conclude that the horizontal distribution is also integral and its leaves are totally geodesic submanifolds in M. Consequently, if M admits a non-trivial submersion with totally geodesic fibers, it cannot be isometrically immersed into any Riemannian manifold of non-positive sectional curvature as a minimal submanifold. □

3. Proof of Theorem 2. Let $\pi\colon M \to B$ be a Riemannian submersion with totally geodesic fibers. Assume that $\phi\colon M \to \mathbf{E}^m$ is an isometric minimal immersion of M into the Euclidean m-space. Then from (2.15) we know that $\breve{A}$ vanishes identically on M and the inequality (2.15) is actually an equality. Thus, we have $A_XV = 0$ for any horizontal vector field X and any vertical vector field V in M. As we already know from the proof of Theorem 1 that this implies that the horizontal distribution $\mathcal{H}$ is an integral distribution and its leaves are totally geodesic. Therefore, M is locally the Riemannian product $F \times B$ of a (totally geodesic) fiber F and the base manifold B.

On the other hand, since the inequality (2.15) is an equality, all of the inequalities in (2.10) and (2.14) become equalities. Hence, the second fundamental form h of M in $\mathbf{E}^m$ satisfies the following two conditions:

$$\sum_{i=1}^{b} h(e_i,e_i) = \sum_{s=b+1}^{n} h(e_s,e_s), \tag{3.1}$$

$$h(X,V)=0,\ \ \forall X\in\mathcal{H},\ \forall V\in\mathcal{V}, \tag{3.2}$$

where $e_1,\dots,e_b$ and $e_{b+1},\dots,e_n$ are orthonormal horizontal and vertical frames, respectively.

Now, it follows from (3.2) and Moore's lemma [9] that $\phi\colon F\times B\to\mathbf{E}^m$ is locally the product immersion of two isometric immersions $\phi_F\colon F\to\mathbf{E}^{m_1}$ and $\phi_B\colon B\to\mathbf{E}^{m_2}$. Since ϕ is assumed to be a minimal isometric immersion, condition (3.1) implies that both ϕ_F and ϕ_B are minimal isometric immersions as well. □

4. Applications. If the target manifold in Theorem 1 is of constant sectional curvature, then we have.

Corollary 1. *Let $\pi\colon M\to B$ be a Riemannian submersion with totally geodesic fibers. Then, for any isometric immersion of M into a Riemannian m-manifold $R^m(\epsilon)$ of constant sectional curvature ϵ, the submersion invariant $\breve{A}_\pi$ on M satisfies*

$$\breve{A}_\pi \le \frac{n^2}{4}H^2 + b(n-b)\epsilon. \tag{4.1}$$

If the target manifold is negatively curved, we obtain.

Corollary 2. *If a Riemannian manifold admits a Riemannian submersion with totally geodesic fibers, then it cannot be isometrically immersed in any Riemannian manifold of negative sectional curvature as a minimal submanifold.*

Corollary 3. *Every Riemannian manifold which admits a non-trivial Riemannian submersion with totally geodesic fibers cannot be isometrically immersed in any Hermitian symmetric space of non-compact type as a minimal submanifold.*

These three corollaries follows easily from (2.15).

Remark 1. When the Riemannian manifold M is a Riemannian product and the target manifold $\tilde{M}$ is of constant negative curvature, Corollary 2 is due to N. Ejiri [7].

Remark 2. The results obtained above can be applied to various very large families of Riemannian manifolds, since Riemannian submersions with totally geodesic fibers occur widely in geometry.

For examples, we have:

(i) The well-known Hopf fibrations:

$$\pi\colon S^{2n+1}\to CP^n(4) \quad\text{and}\quad \pi\colon S^{4n+3}\to HP^n(4)$$

are Riemannian submersions with totally geodesic fibers.

(ii) Let $\pi\colon M\to B$ be a Riemannian submersion with totally geodesic fibers. If B' is a submanifold of B, then the restriction of π to $\pi^{-1}(B')$:

$$\pi\colon \pi^{-1}(B')\to B'$$

is a Riemannian submersion with totally geodesic fibers.

For instance, for any submanifold N of the complex projective n-space $CP^n(4)$ of constant holomorphic sectional curvature 4, $\pi\colon\pi^{-1}(N)\to N$ is a Riemannian submersion with totally geodesic fibers. For this submersion, the invariant $\breve{A}_\pi$ is given by

$$\breve{A}_\pi = \|P\|^2, \tag{4.2}$$

where $P\colon\mathcal{H}\to\mathcal{H}$ is the endomorphism such that PX is the projection of ϕX onto $\mathcal{H}$, ϕ being the $(1,1)$-tensor of the natural Sasakian structure on S^{2n+1}.

(iii) If G is a Lie group equipped with a bi-invariant Riemannian metric and H is a closed subgroup, then the usual Riemannian structure on the homogeneous space G/H is characterized by the fact that the natural mapping $\pi\colon G\to G/H$ is a Riemannian submersion.

The fibers of such a submersion are the left cosets of G (mod H) which are totally geodesic. The invariant $\breve{A}_\pi$ is given by

(4.3)
$$\breve{A}_\pi = \frac{1}{4}\sum_{i,j=1}^{b}\sum_{b+1}^{n}\langle[e_i,e_j],e_s\rangle^2, \quad b=\dim H,$$

where $e_1,\dots,e_b$ are orthonormal left-invariant horizontal vector fields and $e_{b+1},\dots,e_n$ an orthonormal basis of the vertical distribution $\mathcal{V}$.

(iv) The frame bundle $F(B)$ of a b-dimensional Riemannian manifold B is a principal bundle over B with structure group $O(b)$. There exists a natural Riemannian structure on $F(B)$ such that the projection: $\pi\colon F(B)\to B$ is a Riemannian submersion with totally geodesic fibers (see [11]).

(v) If (M, J, g) is an almost Hermitian manifold, its tangent bundle $T(M)$ is also an almost Hermitian manifold with almost Hermitian structure (J^H, g_s), where J^H is the horizontal lift of J and g_s is the Sasaki metric given by:

$$g_s(X^H, Y^H) = g_s(X^V, Y^V) = (g(X,Y))^V$$

and $g_s(X^H, Y^V) = 0$, where X^H and Y^V denote the horizontal and vertical lifts of X and Y, respectively (cf. [12]). The projection:

$$\pi\colon (T(M), J^H, g_s) \to (M, J, g)$$

is an almost Hermitian submersion with totally geodesic fibers.

(vi) On an oriented Riemannian 4-manifold N, there exists an S^2-bundle Z, called the twistor space of N, whose fiber over any point $x \in N$ consists of all almost complex structures on T_xN that are compatible with the metric and the orientation. It is known that there is one-parameter family of metrics g^t on Z, making the projection $Z \to N$ into a Riemannian submersion with totally geodesic fibers.

Remark 3. For a Kaehler submanifold N of $CP^m(4)$, the pre-image $\pi^{-1}(N)$ via the Hopf fibration is a minimal submanifold of S^{2m+1} satisfying the equality case of (4.1) with $n = 1 + \dim_{\mathbf{R}} N$ and $\epsilon = 1$. Thus, the equality case of (4.1) is achieved by many examples. For the classification of Riemannian submersions satisfying the equality case, see [5].

5. A remark on Theorem 1. Theorem 1 is sharp. This can be seen as follows:

(a) The product immersion of the two minimal isometric immersions in Euclidean spaces given in (1.1) shows that the "*non-triviality condition* imposed on the Riemannian submersions cannot be omitted from Theorem 1.

(b) The condition "*non-positive sectional curvature*" imposed on the target manifold also cannot be omitted. For instance, the Hopf fibration: $\pi\colon S^{2n+1} \to CP^n(4)$ is a non-trivial Riemannian submersion with totally geodesic fibers. Clearly, S^{2n+1} can be imbedded as a totally geodesic hypersurface in S^{2n+2}.

(c) The condition "*Riemannian submersion has totally geodesic fibers*" cannot be omitted from Theorem 1 as well, since there exist Euclidean minimal submanifolds which admit non-trivial Riemannian submersions. The simplest such examples are the catenoid $\mathcal{C}$ and the helicoids $\mathcal{H}_a$, $a > 0$, in $\mathbf{E}^3$.

The catenoid $\mathcal{C}$ is defined by

$$(5.1) \qquad (\cosh u \cos v, \cosh u \sin v, u).$$

The catenoid is a minimal surface which admits a non-trivial submersion $\pi\colon \mathcal{C} \to B$ such that the manifold B is the profile curve and the projection $\pi\colon \mathcal{C} \to B$ is the mapping which carries $(\cosh u \cos v, \cosh u \sin v, u) \in \mathcal{C}$ to $(\cosh u, u) \in B$. Fibers of this submersion are the circles of latitude.

For each positive number a, the helicoid $\mathcal{H}_a$ is defined by $(t\cos s, t\sin s, as)$ for $t > 0$. The helicoid is a minimal surface which admits a non-trivial submersion $\pi\colon \mathcal{H}_a \to L_+$, where L_+ is the half line $\{t\colon t > 0\}$ and the projection $\pi\colon \mathcal{H}_a \to L_+$ carries $(t\cos s, t\sin s, as) \in \mathcal{H}_a$ to $t \in L_+$. Fibers of this submersion are helices.

6. A cohomology class. Now, we define a cohomology class, denoted by $c_\pi(M)$, associated with each Riemannian submersion $\pi\colon M \to B$ with orientable base manifold B as follows:

Let $b = \dim B$, $n = \dim M$, and let $e_1, \dots, e_n$ be a local orthonormal frame on M which satisfies the following two conditions:

(i) $e_{b+1}, \dots, e_n$ are vertical vector fields and

(ii) $e_1, \dots, e_b$ are basic horizontal vector fields such that $(e_1)_*, \dots, (e_b)_*$ gives rise to the positive orientation of B.

Let $\omega^1, \dots, \omega^n$ be the dual frame of $e_1, \dots, e_n$. Consider the b-form ω on M defined by

$$(6.1) \qquad \omega = \omega^1 \wedge \cdots \wedge \omega^b.$$

Then we have $d\omega = 0$, since ω is the pull back of the volume form of B. Thus ω defines a cohomology class $c_\pi(M) = [\omega] \in H^b(M; \mathbf{R})$.

Theorem 3. *Let $b = \dim B$ and $\pi\colon M \to B$ be a Riemannian submersion with minimal fibers and orientable base manifold B. If M is a closed manifold with $H^b(M; \mathbf{R}) = 0$, then the horizontal distribution $\mathcal{H}$ of the Riemannian submersion is never integrable. Thus the submersion is always non-trivial.*

Since each nonzero harmonic form represents a non-trivial cohomology class, Theorem 3 follows from the following

Theorem 4. *Let $\pi\colon M \to B$ be a Riemannian submersion from a closed manifold M onto an orientable base manifold B. Then the pull back of the volume element of B is harmonic if and only if the horizontal distribution $\mathcal{H}$ is integrable and fibers are minimal.*

Proof. Let $e_1, \dots, e_n$ and $\omega^1, \dots, \omega^n$ be defined as above. Then we have

$$\omega^j(e_s) = 0, \;\; \omega^i(e_j) = \delta_{ij} \tag{6.2}$$

for $1 \leq i, j \leq b$; $b + 1 \leq s \leq n$.

Let us put

$$\omega^{\perp} = \omega^{b+1} \wedge \dots \wedge \omega^n. \tag{6.3}$$

The we have

$$d\omega^{\perp} = \sum_{i=b+1}^{n} (-1)^i \omega^{b+1} \wedge \dots \wedge d\omega^{b+i} \wedge \dots \wedge \omega^n. \tag{6.4}$$

It follows from (6.2) and (6.4) that $d\omega^{\perp} = 0$ holds if and only if the following two conditions hold:

$$d\omega^{\perp}(X, Y, V_1, \dots, V_{n-b-1}) = 0 \tag{6.5}$$

for horizontal vector fields X, Y and vertical vector fields $V_1, \dots, V_{n-b-1}$; and for $s = 1, \dots, b$ we have

$$d\omega^{\perp}(e_s, e_{b+1}, \dots, e_n) = 0. \tag{6.6}$$

From (6.4) and (6.3) we find

$$\begin{aligned} d\omega^{\perp}&(X, Y, V_1, \dots, V_{n-b-1}) \\ &= \omega^{\perp}([X, Y], V_1, \dots, V_{n-b-1}). \end{aligned} \tag{6.7}$$

It is easy to see from (6.7) that condition (6.5) holds if and only if the horizontal distribution $\mathcal{H}$ of the Riemannian submersion is integrable.

From (6.2) and (6.3) we also have

$$\begin{aligned} &d\omega^{\perp}(e_s, e_{b+1}, \dots, e_n) \\ &= \sum_{i=1}^{n-b} (-1)^{1+i} \omega^{\perp}([e_s, e_{b+i}], e_{b+1}, \dots, \widehat{e}_{b+i}, \dots, e_n) \\ &= \sum_{i=1}^{n-b} \left\{ \omega^{b+i}(\nabla_{e_s} e_{b+i}) - \omega^{b+i}(\nabla_{e_{b+i}} e_s) \right\} \\ &= -\sum_{i=1}^{n-b} \langle \nabla_{e_{b+i}} e_s, e_{b+i} \rangle \\ &= \sum_{i=1}^{n-b} \langle A_{e_{b+i}} e_{b+i}, e_s \rangle \end{aligned}$$

for each $s \in \{1, \dots, b\}$. Hence, condition (6.6) holds if and only if fibers of the Riemannian submersion are minimal. Consequently, the pull back of the volume element of B is co-closed (or equivalently, $d\omega^{\perp} = 0$ holds) if and only if the horizontal distribution $\mathcal{H}$ of the Riemannian submersion is integrable and fibers are minimal. □

Remark 4. A cohomology class similar to $c_\pi(M)$ for CR submanifolds of a Kaehler manifold has been introduced earlier in [2]. The proof of Theorem 4 bases on the same idea as that in [2].

Remark 5. It is known that the minimality of fibers is equivalent to the harmonicity of the submersion [6]. The condition for the integrability of the horizontal distribution and the minimality of fibers for a Riemannian submersion $\pi\colon M \to B$ were studied in [8] in view of the commutativity of the Laplacian acting on p-forms on B and the Laplacian acting on p-forms on M with $b \geq p \geq 1$.

Acknowledgement. The author thanks the referee for several valuable suggestions.

References

[1] B.-Y. Chen, *Geometry of submanifolds*, Dekker, New York, 1973.
[2] B.-Y. Chen, *Geometry of submanifolds and its applications*, Sci. Univ. Tokyo, Tokyo, 1981.
[3] B.-Y. Chen, Some pinching and classification theorems for minimal submanifolds, Arch. Math. (Basel) **60** (1993), no. 6, 568–578.
[4] B.-Y. Chen, Some new obstructions to minimal and Lagrangian isometric immersions, Japan. J. Math. (N.S.) **26** (2000), no. 1, 105–127.
[5] B.-Y. Chen, Examples and classification of Riemannian submersions satisfying a basic equality. Bull. Austral. Math. Soc. **72** (2005), 391–402.
[6] J. Eells, Jr. and J. H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. **86** (1964), 109–160.
[7] N. Ejiri, Minimal immersions of Riemannian products into real space forms, Tokyo J. Math. **2** (1979), no. 1, 63–70.
[8] P. B. Gilkey, J. V. Leahy and J. Park, *Spinors, spectral geometry, and Riemannian submersions*, Seoul Nat. Univ., Seoul, 1998.
[9] J. D. Moore, Isometric immersions of riemannian products, J. Differential Geometry **5** (1971), 159–168.
[10] T. Nagano, On fibred Riemann manifolds, Sci. Papers Coll. Gen. Ed. Univ. Tokyo **10** (1960), 17–27.
[11] B. O'Neill, The fundamental equations of a submersion, Michigan Math. J. **13** (1966), 459–469.
[12] K. Yano and S. Ishihara, *Tangent and cotangent bundles*: *differential geometry*, Dekker, New York, 1973.