Strong symplectic structures on spaces of probability measures with positive density function

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Abstract: Spaces of probability measures with positive density function on a compact Riemannian manifold are endowed with a closed 2-form associated with the Fisher information metric by using a divergence-free vector field. In this note we give a necessary and sufficient condition on the vector field that this 2-form is a *strong* symplectic structure.

Key words: Fisher information metric; symplectic structure; Hilbert manifold; probability measure.

1. Introduction. The aim of this article is to study a basic question concerning symplectic structures on an infinite dimensional manifold.

On a finite dimensional smooth manifold M a symplectic structure is defined by giving a closed 2-form ω which is non-degenerate. Here ω is called non-degenerate when at every point x of M $\omega_x(X, Y) = 0$ for all tangent vectors Y implies $X = 0$. The 2-form ω induces a linear mapping $\tilde{\omega}_x$ of T_xM to T_x^*M, $x \in M$, and the non-degeneracy of ω is equivalent to the strong non-degeneracy of ω, i.e., $\tilde{\omega}_x$ being isomorphic at every point. However, on an infinite dimensional manifold the non-degeneracy of a closed 2-form ω does not implies, in general, the strong non-degeneracy of ω. Refer to [8] for the notion of strong symplectic structure.

We say that a symplectic manifold (M, ω) is strong if ω is strongly symplectic. In symplectic geometry the classical theorem of Darboux plays an essential role. For a strong symplectic manifold (M, ω) the theorem of Darboux holds even though the manifold M has infinite dimension. See for this [11, 12].

Though many infinite dimensional symplectic manifolds have been encountered in mathematics, unfortunately few examples of infinite dimensional symplectic manifold which is strong are known. In this sense, it is of significance to find infinite dimensional strong symplectic manifolds.

2. Spaces of probability measures with positive density function and the Fisher information metric. Let (X, h) be a compact, oriented, n-dimensional Riemannian manifold with a Riemannian metric h. Let dV_h be the canonical volume element defined by

$$dV_h = \sqrt{\det(h_{ij})}\, dx^1 \wedge \cdots \wedge dx^n,$$

where $(x^1, \ldots, x^n)$ is a local coordinate and $h_{ij} = h(\partial/\partial x^i, \partial/\partial x^j)$. We assume X has unit volume, that is, $\int_X dV_h = 1$. For a fixed positive integer k consider a space of probability measures whose density function is in $L_k^2(X)$ and positive everywhere;

$$\mathcal{P}_k(X) = \left\{ \mu = f dV_h \;\middle|\; f \in L_k^2(X),\ f > 0,\ \int_X \mu = 1 \right\},$$

where $L_k^2(X)$ is the Sobolev space consisting of functions on X of finite L_k^2-norm. Here the L_k^2-norm $|\varphi|^2_{L_k^2}$ is given by the inner product

$$\langle \varphi, \psi \rangle = \int_X \varphi\psi\, dV_h + \sum_{j=1}^{k} \int_X h(\nabla^j \varphi, \nabla^j \psi)\, dV_h.$$

Choose k as $k > 1+n/2$ and identify the space $\mathcal{P}_k(X)$ with $Q_k(X) = \left\{ f \in L_k^2(X) \;\middle|\; \int_X f\, dV_h = 1,\ f > 0 \right\}$, an open subset in the closed affine subspace $U_k(X) = \left\{ f \in L_k^2(X) \;\middle|\; \int_X f\, dV_h = 1 \right\}$ of the space $L_k^2(X)$. From the Sobolev embedding theorem ([2]) it is easily seen that $Q_k(X)$ is open. Via this identification we can define a topology on $\mathcal{P}_k(X)$. $\mathcal{P}_k(X)$ has a Hilbert manifold structure modeled on the Hilbert space $\left\{ u \in L_k^2(X) \;\middle|\; \int_X u\, dV_h = 0 \right\}$. For a detailed argument refer to [10]. Furthermore for a basic reference of Hilbert manifolds see [6].

In 1945 C.R. Rao pointed out that the Fisher information matrix determines a Riemannian metric on a space of probability measures, and then infor-

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mation geometry was established and has made a great contribution to study of statistical inference. For this refer to [1].

The *Fisher information metric* g is a Riemannian metric on $\mathcal{P}_k(X)$ defined by

$$g(\sigma_1, \sigma_2) = \int_X \frac{d\sigma_1}{d\mu}\frac{d\sigma_2}{d\mu}\mu$$

for $\mu \in \mathcal{P}_k(X)$, $\sigma_1, \sigma_2 \in T_\mu\mathcal{P}_k(X)$, where $d\sigma_i/d\mu$ denotes the density function of σ_i with respect to μ ($i = 1, 2$). Note that g is nondegenerate, since the density function of $\mu \in \mathcal{P}_k(X)$ is strictly positive over X. The metric g can be considered as an infinite dimensional version of the Fisher information matrix.

To construct a symplectic structure on $\mathcal{P}_k(X)$ we provide a divergence-free smooth vector field W on (X, h). We define a linear operator $\Phi\colon T_\mu\mathcal{P}_{k+1}(X) \to T_\mu\mathcal{P}_k(X)$ as

$$\Phi(\sigma) = W\left(\frac{d\sigma}{d\mu}\right) dV_h.$$

We can consider Φ as a differential operator on $T_\mu\mathcal{P}_k(X)$ by virtue of the fact that $T_\mu\mathcal{P}_{k+1}(X)$ is dense in $T_\mu\mathcal{P}_k(X)$. Since Φ is an anti-symmetric operator with respect to g ([4, 10]), we can define a 2-form Ω on $\mathcal{P}_k(X)$ as;

$$\begin{aligned}\Omega(\sigma_1, \sigma_2) &= g(\sigma_1, \Phi(\sigma_2)) \\ &= \int_X \frac{d\sigma_1}{d\mu} W\left(\frac{d\sigma_2}{d\mu}\right) dV_h.\end{aligned}$$

In [4], T. Friedrich gave a condition that the 2-form Ω is a symplectic structure.

Theorem 1 (T. Friedrich)**.** *The 2-form Ω is closed. Furthermore if W has an integral curve which is dense in X, then Ω is a symplectic structure on $\mathcal{P}_k(X)$.*

However, we can improve the theorem of Friedrich as follows:

Theorem 2. *The 2-form Ω is symplectic if and only if the vector field W has no regular first integral.*

A function f on a manifold M is called a *regular first integral* of a vector field W if f satisfies $Wf = 0$ and is not constant on any open subset of M.

Let $\mathrm{Diff}_k(X)$ be the group of orientation preserving L^2_k-diffeomorphisms of X. A map $\varphi\colon X \to X$ is an L^2_k-map when every coordinate representation of φ $\psi_V^{-1} \circ \varphi \circ \psi_U\colon U \subset \mathbf{R}^n \to V \subset \mathbf{R}^n$ is in $L^2_k(U, \mathbf{R}^n)$. We call a bijective map $\varphi\colon X \to X$ an L^2_k-diffeomorphism when φ and φ^{-1} are L^2_k-maps. Since $\mathrm{Diff}_k(X)$ is open in the space of L^2_k-maps for $k > 1+n/2$ (see [7], p. 88), $\mathrm{Diff}_k(X)$ admits a Hilbert manifold structure modeled on a Hilbert space.

Let $\mathcal{P}_\infty(X)$ be the space of probability measures whose density function is smooth and strictly positive. Then $\mathrm{Diff}(X)$ acts on $\mathcal{P}_\infty(X)$ by pull-back, where $\mathrm{Diff}(X)$ is the group of orientation preserving, smooth diffeomorphisms. This action extends to a map $\mathrm{Diff}_{k+1}(X) \times \mathcal{P}_k(X) \to \mathcal{P}_k(X)$ (see [3], p. 38). From the theorem of Moser for Sobolev volume elements ([3], p. 38, Prop. 8.12) the action is transitive. As is stated in [4], this action is isometric with respect to the Fisher information metric.

Let $\mathcal{D}^\mu_{k+1}$ be the group of diffeomorphisms fixing μ. Note that the map $\varphi \mapsto \varphi^*(\mu)$ gives a homeomorphism $\mathrm{Diff}_{k+1}(X)/\mathcal{D}^\mu_{k+1} \to \mathcal{P}_k(X)$ ([3], p. 38, Prop. 8.13).

Remark. Since the action of $\mathrm{Diff}_{k+1}(X)$ is isometric, $\varphi \in \mathrm{Diff}_{k+1}(X)$ preserves the symplectic form Ω if and only if φ preserves the $(n-1)$-form $i_W dV_h$. Refer to [4, 9, 10] for this statement in terms of the Poisson structure.

3. Strong symplectic structures. Our problem is to find a condition which guarantees that the mapping $\tilde{\Omega}\colon T_\mu\mathcal{P}_k(X) \to (T_\mu\mathcal{P}_k(X))^*$ induced from Ω is isomorphic.

Theorem 3. *Let W be a smooth divergence-free vector field without regular first integrals. Then, the symplectic manifold $(\mathcal{P}_k(X), \Omega)$ is strong if and only if Φ is an isomorphism.*

Proof. Assume Φ is isomorphic. Fix $\mu \in \mathcal{P}_k(X)$. Since Ω is a symplectic structure, $\tilde{\Omega}$ is injective. Hence it suffices to prove that $\tilde{\Omega}$ is surjective. Let $\tau \in (T_\mu\mathcal{P}_k(X))^*$. From the nondegeneracy of the Fisher information metric there exists $\tau^\sharp \in T_\mu\mathcal{P}_k(X)$ such that

$$\tau(\sigma) = g(\tau^\sharp, \sigma)$$

for all $\sigma \in T_\mu\mathcal{P}_k(X)$. Since the operator Φ is onto, there exists $\rho \in T_\mu\mathcal{P}_k(X)$ such that $\Phi(\rho) = \tau^\sharp$. Thus, we have for all $\sigma \in T_\mu\mathcal{P}_k(X)$

$$\tau(\sigma) = g(\Phi(\rho), \sigma) = \Omega(\rho, \sigma),$$

which implies the surjectivity of $\tilde{\Omega}$.

Conversely we assume the surjectivity of $\tilde{\Omega}$. Take $\tau^\sharp \in T_\mu\mathcal{P}_k(X)$ such that $g(\tau^\sharp, \sigma) = \tau(\sigma)$ for $\tau \in (T_\mu\mathcal{P}_k(X))^*$ and all $\sigma \in T_\mu\mathcal{P}_k(X)$. Then there exists $\rho \in T_\mu\mathcal{P}_k(X)$ such that $\Omega(\rho, \sigma) = \tau(\sigma)$ for all σ. Making use of the definition of Ω, we obtain that $g(\Phi(\rho), \sigma) = g(\tau^\sharp, \sigma)$ for all σ. Using the non-degeneracy of the Fisher information metric again, it follows that Φ is onto. □

We end this note by giving an example of the space of probability measures admitting a strong symplectic structure.

Let S^1 be a unit circle with the angular coordinate t. Take $1/2\pi\, dt$ as the canonical probability measure. The space $\mathcal{P}_k(S^1)$ of probability measures on S^1 is a homogeneous space with natural action of $\mathrm{Diff}_{k+1}(S^1)$ for $k \geq 2$. Since the isotropy subgroup of the measure $1/2\pi\, dt$ is the rotation group, isomorphic to S^1, $\mathcal{P}_k(S^1)$ is homeomorphic to the coset space $\mathrm{Diff}_{k+1}(S^1)/S^1$ by the remark of Section 2 (see [5, 9] for reference). The space $\mathrm{Diff}(S^1)/S^1$ is an object of importance in the string theory in physics. In [5], Kirillov and Yur'ev studied an infinite dimensional Kähler geometry of $\mathrm{Diff}(S^1)/S^1$.

The space of divergence-free vector fields on S^1 is spanned by the coordinate vector field d/dt. Obviously d/dt has no regular first integrals, so the 2-form Ω on $\mathcal{P}_k(S^1)$ is a symplectic structure, when we take $W = d/dt$. Moreover the linear operator Φ is surjective. This can be shown as follows: Since $\mathcal{P}_k(S^1)$ is a homogeneous space of $\mathrm{Diff}_{k+1}(S^1)$, it suffices to prove this assertion at some point of $\mathcal{P}_k(S^1)$. Set $\mu = 1/2\pi\, dt$ and let $\sigma = f\mu$ be in $T_\mu\mathcal{P}_k(S^1)$. Then we can find $\rho \in T_\mu\mathcal{P}_k(S^1)$ such that $\Phi(\rho) = \sigma$ as follows: Since each element of the tangent space $T_\mu\mathcal{P}_k(S^1)$ is an L^2-integrable function f of period 2π on $\mathbf{R}$ with $\int_0^{2\pi} f dt = 0$, we have that $T_\mu\mathcal{P}_k(S^1)$ is spanned by $\sin nt$, $\cos nt$ $(n = 1, 2, 3, \ldots)$. Hence for $\sigma = f\mu$ we can set $f = \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$. Put $h = \sum_{n=1}^{\infty}(-b_n/n \cos nt + a_n/n \sin nt)$ and define $\rho = h\mu \in T_\mu\mathcal{P}_k(S^1)$. Then $\Phi(\rho) = \sigma$, since $dh/dt = f$. Therefore Φ is surjective so that from our theorem Ω is strong. Since $i_W(\mu) = 1/2\pi$ is constant function, from the remark of the previous section every element of $\mathrm{Diff}_{k+1}(S^1)$ preserves the symplectic form Ω which is strong.

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References

[1] S. Amari and H. Nagaoka, *Methods of information geometry*, Translated from the 1993 Japanese original by Daishi Harada, Amer. Math. Soc., Providence, RI, 2000.

[2] T. Aubin, *Nonlinear analysis on manifolds. Monge-Ampère equations*, Springer, New York, 1982.

[3] D.G. Ebin, The manifold of Riemannian metrics, in *Global Analysis* (*Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968*), 11–40, Amer. Math. Soc., Providence, R.I.

[4] T. Friedrich, Die Fisher-Information und symplektische Strukturen, Math. Nachr. **153** (1991), 273–296.

[5] A.A. Kirillov, and D.V. Yur'ev, Kähler geometry of the infinite-dimensional homogeneous space $M = \mathrm{Diff}_+(S^1)/\mathrm{Rot}(S^1)$, Funktsional. Anal. i Prilozhen. **21** (1987), no. 4, 35–46, 96.

[6] S. Lang, *Differential manifolds*, Addison-Wesley, 1972.

[7] J. Marsden, *Applications of global analysis in mathematical physics*, Publish or Perish, Boston, Mass., 1974.

[8] J.E. Marsden and T.S. Ratiu, *Introduction to mechanics and symmetry*, Springer, New York, 1994.

[9] Y. Shishido, Geometry of infinite dimensional statistical manifolds, master thesis, (2004).

[10] Y. Shishido, A note on the Poisson structures of the space of probability measures. (Submitted).

[11] A. Weinstein, Symplectic structures on Banach manifolds, Bull. Amer. Math. Soc. **75** (1969), 1040–1041.

[12] A. Weinstein, Symplectic manifolds and their Lagrangian submanifolds, Advances in Math. **6** (1971), 329–346.