Dependance of Dirichlet integrals upon lumps of Riemann surfaces

By Mitsuru NAKAI*)

Professor Emeritus, Nagoya Institute of Technology

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Abstract: Take a simple arc γ in an open Riemann suface R carrying a nonconstant harmonic function u with finite Dirichlet integral $D(u;R)$. Form a Riemann surface R_γ with lump $\widehat{\mathbf{C}}\setminus\gamma$ by pasting $R\setminus\gamma$ with $\widehat{\mathbf{C}}\setminus\gamma$ crosswise along γ, i.e. $R_\gamma := (R\setminus\gamma)\bowtie_\gamma(\widehat{\mathbf{C}}\setminus\gamma)$, and the transplant u_γ of u on R to R_γ characterized by its being harmonic on R_γ with $D(u_\gamma;R_\gamma) < +\infty$ and $u_\gamma = u$ at the ideal boundary of R_γ and hence of R in a suitable sense. We are interested in the comparison of $D(u_\gamma;R_\gamma)$ with $D(u;R)$ when we take a variety of choices of pasting arcs γ in R, and we will prove that $D(u_\gamma;R_\gamma) < D(u;R)$ for any u level arc γ in R, $D(u_\gamma;R_\gamma) > D(u;R)$ for any u conjugate level arc γ in R, and as a consequence of these two facts there is a nondegenerate arc γ (i.e. not a point arc γ) in R such that $D(u_\gamma;R_\gamma) = D(u;R)$.

Key words: Conjugate level arc; Dirichlet integral; level arc; pasting arc; Riemann surface with lump; Royden decomposition.

Take a simple arc γ in a Riemann surface R. Since γ is simple we can embed γ conformally in the complex plane $\mathbf{C}$ so that we can view R and $\widehat{\mathbf{C}}$ have the arc γ in common. We form a new Riemann surface R_γ by pasting $R\setminus\gamma$ with $\widehat{\mathbf{C}}\setminus\gamma$ crosswise along γ, where $\widehat{\mathbf{C}} = \mathbf{C}\cup\{\infty\}$ is the complex sphere with ∞ the point at infinity of $\mathbf{C}$. We have been using the following impressive notation for R_γ:

$$R_\gamma := (R\setminus\gamma)\bowtie_\gamma(\widehat{\mathbf{C}}\setminus\gamma).$$

The surface R_γ will be referred to as a Riemann surface with a *lump* $\widehat{\mathbf{C}}\setminus\gamma$ obtained from R by hitting R at γ.

The *Dirichlet integral* $D(f;R)$ of a real valued function f in $W^{1,2}_{\mathrm{loc}}(R)$, the local Sobolev space on R, over R is the quantity given by

$$D(f;R) := \int_R df\wedge *df.$$

We denote by $L^{1,2}(R)$ the *Dirichlet space* (cf. [3]) which is the class of functions $f\in W^{1,2}_{loc}(R)$ with finite Dirichlet integrals $D(f;R) < +\infty$ over R. For two functions f and g in $L^{1,2}(R)$ we can consider the quantity

$$D(f,g;R) := \int_R df\wedge *dg.$$

This is a convenient tool for the computation of Dirichlet integrals and is referred to as the *mutual* Dirichlet integral of f and g over R.

It is traditional in the classification theory of Riemann surfaces (cf. e.g. [5]) to use the notation $HD(R)$ for the class of harmonic functions u on R with finite Dirichlet integrals $D(u;R)$ taken over R and O_{HD} the class of Riemann surfaces R such that $HD(R)$ is trivial, i.e. $HD(R) = \mathbf{R}$ (the set of real numbers). It is known that $O_G < O_{HD}$ (strict inclusion), where O_G is the class of parabolic (i.e. not hyperbolic) Riemann surfaces characterized by the nonexistence of Green functions on them. Hence, as far as we require for a Riemann surface R to have a nontrivial harmonic function with finite Dirichlet integral we must assume first of all that R is hyperbolic. For such surfaces R, we set

$$\mathcal{D}(R) := L^{1,2}(R)\cap C(R),\ \mathcal{D}_0(R) := L^{1,2}(R)\cap C_0(R),$$

where $C_0(R)$ is the class of $f\in C(R)$ with compact supports in R, and finally we denote by

$$\mathcal{D}_\Delta(R)$$

the class of $f\in\mathcal{D}(R)$ such that there exists a sequence $(f_n)_{n\geq 1}$ in $\mathcal{D}_0(R)$ converging to f almost uniformly on R (i.e. uniformly on each compact subset of R) and at the same time $D(f-f_n;R)\to 0$ $(n\to$

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*) Present address: Eguchi 52, Hinaga, Chita, Aichi 478-0041.

∞). Clearly $\mathcal{D}_0(R) \subset \mathcal{D}_\Delta(R) \subset \mathcal{D}(R)$ and $HD(R) \subset \mathcal{D}(R)$. The function $f \in \mathcal{D}_\Delta(R)$ is referred to as a *Dirichlet potential* on R since $f \in \mathcal{D}_\Delta(R)$ is characterized as a function $f \in \mathcal{D}(R)$ such that there exists a potential (a positive superharmonic function with the vanishing greatest harmonic minorant on R) p_f on R with $|f| \le p_f$ on R (cf. e.g. [2]). Therefore we may impressively say that a function f in $\mathcal{D}_\Delta(R)$ vanishes at the ideal boundary of R and similarly, for two functions f and g in $\mathcal{D}(R)$, f equals g at the ideal boundary of R if $f - g \in \mathcal{D}_\Delta(R)$.

Fix a compact subset K of R with connected complement $R \setminus K$. We say that a subregion Ω of R whose relative boundary $\partial\Omega$ of Ω consists of a finite number of mutually disjoint piecewise smooth Jordan curves is a *smooth ideal boundary neighborhood* of R excluding K if $R \setminus \Omega$ is compact and $K \subset R \setminus \overline{\Omega}$. Suppose we have two functions $f \in \mathcal{D}_\Delta(R \setminus K)$ and $g \in \mathcal{D}(R \setminus K)$. Then we have the following consequence (cf. [5]) of the Green formula if moreover the second function g is supposed to be smooth in a vicinity of $\partial\Omega$:

$$(1) \qquad \int_\Omega df \wedge *dg + \int_\Omega f d * dg = \int_{\partial\Omega} f * dg.$$

Here formally in general we have to add something like the term $\int_\delta f * dg$ on the right hand side of the above identity (1) to obtain the complete Green formula, where δ is the ideal boundary of Ω (so that of R) but, since $f = 0$ on δ, we can disregard this term. This is the intuitive explanation of the significance of (1).

We have the following direct sum decomposition of $\mathcal{D}(R)$ (cf. e.g. [2] and [5]), which is referred to as the *Royden decomposition* of $\mathcal{D}(R)$:

$$(2)_1 \qquad \mathcal{D}(R) = HD(R) + \mathcal{D}_\Delta(R)$$

with $HD(R) \cap \mathcal{D}_\Delta(R) = \{0\}$ in the sense that every $f \in \mathcal{D}(R)$ can be uniquely expressed as $f = u + g$ with $u \in HD(R)$ and $g \in \mathcal{D}_\Delta(R)$ satisfying the *Dirichlet principle*:

$$(2)_2 \qquad D(f;R) = D(u;R) + D(g;R).$$

The function u in the Royden decomposition $f = u + g$ is referred to as the *harmonic part* of the Royden decomposition of f.

Now we assume that there is nontrivial harmonic function u on R with finite Dirichlet integral $D(u;R)$. For a simple arc $\gamma \subset R$ form the Riemann surface R_γ with lump $\widehat{\mathbf{C}} \setminus \gamma$ by hitting R at γ. It is easy to find an $f \in \mathcal{D}(R_\gamma)$ such that $f = u$ on a smooth ideal boundary neighborhood Ω excluding γ so that $\overline{\Omega} \subset R \setminus \gamma \subset R_\gamma$ and $R_\gamma \setminus \overline{\Omega}$ contains the closure of $\mathbf{C} \setminus \gamma$ in R_γ. Let u_γ be the harmonic part of f which is determined only by u and γ not depending on the particular choice of f. We say that u_γ is the *transplant* of u on R to R_γ. Observe that γ in R_γ gives rise to a Jordan curve α in R_γ such that the relative boundary $\partial(R \setminus \gamma)$ considered in R_γ is α and the relative boundary $\partial(\widehat{\mathbf{C}} \setminus \gamma)$ also considered in R_γ is $-\alpha$. If $\gamma \subset R$ is piecewise smooth, then so is α and (1) takes the form

$$(3) \qquad D(u_\gamma - u, u; R \setminus \gamma) = \int_\alpha (u_\gamma - u) * du.$$

As before we consider R carring a $u \in HD(R) \setminus \mathbf{R}$. A simple arc γ in R is said to be a *u level arc* if $du \wedge *du \ne 0$ *on* γ and $du = 0$ *along* γ, or equivalently, u is a constant on γ. Similarly we say that a simple arc γ in R is a *u conjugate level arc* if $du \wedge *du \ne 0$ *on* γ and $*du = 0$ *along* γ so that any branch of the conjugate harmonic function of u is a constant on γ. We will state and prove the following three theorems. In all of these three theorems we assume that R is a hyperbolic Riemann surface, u is a nonconstant harmonic function on R with the finite Dirichlet integral $D(u;R) < +\infty$ over R, and we denote by u_γ for any simple arc γ on R the transplant of u on R to $R_\gamma = (R \setminus \gamma) \uplus_\gamma (\widehat{\mathbf{C}} \setminus \gamma)$, the Riemann surface with lump obtained from R by hitting R at γ.

Theorem 1. *For any u level arc γ in R the following strict inequality holds:*

$$(4) \qquad D(u_\gamma; R_\gamma) < D(u;R).$$

Proof. Observing that $u = \lambda$ (a constant) on γ, define the new function v on R_γ given by $v = u$ on $R \setminus \gamma$, $v = \lambda$ on $\widehat{\mathbf{C}} \setminus \gamma$, and $v = \lambda$ on γ. Note that u_γ is the harmonic part of v on R_γ and clearly $u_\gamma \ne v$ on R_γ. Therefore by the Dirichlet principle $(2)_2$

$$D(u_\gamma; R_\gamma) < D(v; R_\gamma) = D(v;R) = D(u;R),$$

which is nothing but (4). □

Theorem 2. *For any u conjugate level arc γ in R the following strict inequality holds:*

$$(5) \qquad D(u_\gamma; R_\gamma) > D(u;R).$$

Proof. By (3) and $*du = 0$ along γ and hence along α, we see that

$$D(u_\gamma - u, u; R \setminus \gamma) = \int_\alpha (u_\gamma - u) * du = 0.$$

By the above and the Schwarz inequality

$$D(u; R \setminus \gamma) = D(u_\gamma, u; R \setminus \gamma) \leq D(u_\gamma; R \setminus \gamma)^{1/2} D(u; R \setminus \gamma)^{1/2}$$

and we deduce $D(u; R \setminus \gamma)^{1/2} \leq D(u_\gamma; R \setminus \gamma)^{1/2}$ and a fortiori

$$D(u; R) = D(u; R \setminus \gamma) \leq D(u_\gamma; R \setminus \gamma) < D(u_\gamma; R_\gamma)$$

since $D(u_\gamma; \widehat{\mathbf{C}} \setminus \gamma) > 0$ and thus we can conclude the validity of (5). □

Fix a nonsingular point of u (i.e. a point at which $du \wedge *du$ does not vanish) and a closed parametric disc V: $|z| \leq 1$ centered at the above point such that $du \wedge *du \neq 0$ on V. We denote by $\rho(\zeta)$ the radius of V terminating at $\zeta \in \partial V$. Let $\gamma(\zeta_1)$ ($\gamma(\zeta_2)$, resp.) be the u level arc (u conjugate level arc, resp.) starting from the origin 0, passing through the interior of V, and terminating at $\zeta_1 \in \partial V$ ($\zeta_2 \in \partial V$, resp.) for the first time, and $\gamma(\zeta_1) \cap \gamma(\zeta_2) = \{0\}$. We can have such a situation as described above by taking V small enough if necessary and we may assume the subarc of the circle ∂V bounded by ζ_1 and ζ_2 is $\widehat{\zeta_1\zeta_2} := \{\zeta \in \partial V\colon \arg\zeta_1 \leq \arg\zeta \leq \arg\zeta_2\}$, where $0 \leq \arg\zeta_1 < \arg\zeta_2 < 2\pi$, and we denote by $A := \widehat{\zeta_1\zeta_2} \setminus \{\zeta_1, \zeta_2\}$ so that $\overline{A} = \widehat{\zeta_1\zeta_2}$.

Theorem 3. *While there exist two points ζ_1 and ζ_2 on $\overline{A}$ such that for any arcs γ_1 and γ_2 connecting the origin* 0 *and ζ_1 and ζ_2 respectively contained in the interior of V except for their terminal points*

$$D(u_{\gamma_1}; R_{\gamma_1}) < D(u; R) < D(u_{\gamma_2}; R_{\gamma_2}), \tag{6}$$

there is the third point ζ_3 in A such that for any simple arc γ_3 connecting 0 *and ζ_3 contained in the interior of V except for its terminal point*

$$D(u_{\gamma_3}; R_{\gamma_3}) = D(u; R). \tag{7}$$

Proof. First of all observe that any simple arc γ in V connecting 0 and $\zeta \in \partial V$ and contained in the interior of V except for its end point ζ is homotopic to $\rho = \rho(\zeta)$ in V with a homotopy bridge contained in the interior of V except for terminal points ζ and of course in R and the same is true of γ and ρ in $\mathbf{C}$ if we embed V naturally to $\mathbf{C}$. Hence $R_\gamma = R_\rho$ and thus $u_\gamma = u_\rho$. Therefore (6) is certainly correct in view of Theorems 1 and 2 and the proof of (7) will be over if it is shown for the particular case of $\gamma = \rho(\zeta_3)$. Consider the function

$$d(\zeta) := D(u_{\rho(\zeta)}; R_{\rho(\zeta)}) \qquad (\zeta \in \overline{A})$$

on $\overline{A}$, which is easily seen to be continuous on $\overline{A}$ (cf. [4]) by the standard normal family argument (cf. e.g. [1, 6], etc.). Since $d(\zeta_1) < D(u; R) < d(\zeta_2)$, the intermediate value theorem for continuous functions implies the existence of a $\zeta_3 \in A$ such that $d(\zeta_3) = D(u; R)$. □

By using Theorem 2 above we can complete the proof of the existence of supercritical pasting arcs introduced in [4], where the existence was only established in [4] under an additional technical condition so that we can now remove this unpleasant assumption thanks to our present simple Theorem 2.

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