Note on the Atiyah-Hirzebruch spectral sequence of *KO*-theory

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Abstract: We construct examples of finite complexes without odd cells which have non-trivial E_r-term of the Atiyah-Hirzebruch spectral sequence of KO-theory for $r \geq 3$.

Key words: KO-theory; Atiyah-Hirzebruch spectral sequence.

1. Introduction. KO-theory of various finite complexes without odd cells are computed, such as the Hermitian symmetric spaces, the flag manifolds and so on, by making use of the Atiyah-Hirzebruch spectral sequence [2–5]. In each case it is shown that the spectral sequence collapses for E_3-term by use of the following lemma.

Lemma 1.1. *Let X be a finite complex without odd cells and let $(E_r(X), d_r)$ denote the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$. Then we have:*

(a) $E_3^{p,-1} \cong H^p(H^*(X; \mathbf{Z}/2); Sq^2)$.

(b) *If d_r is the first non trivial differential for $r \geq 3$, then we have:*
   (i) $r \equiv 2\ (8)$
   (ii) *There exists $x \in E_r^{r,0}(X)$ such that $\eta x \neq 0$ and $\eta d_r x \neq 0$, where η is a generator of $KO^{-1}(\mathrm{pt})$.*

Lemma 1.1 is so strong that one might think that for any finite complex without odd cells the Atiyah-Hirzebruch spectral sequence of KO-theory collapses for E_3-term. The purpose of this paper is to construct counter examples of for this account and we obtain:

Theorem 1.1. *For each $r \geq 3$ there exists a finite complex without odd cells, denoted by X, such that $d_r\colon E_r(X) \to E_r(X)$ is non-trivial.*

2. Various *K*-theories. In this section we recall various K-theories which are required in the following section.

We denote the complex, the real and the self conjugate K-theory of a space X by $K^*(X)$, $KO^*(X)$ and $KSC^*(X)$ respectively. KSC-theory is defined by the exact sequence

$$(1)\quad \cdots \to KSC^*(X) \to K^*(X) \xrightarrow{1-\mathbf{t}} K^*(X) \to KSC^{*+1}(X) \to \cdots,$$

where $\mathbf{t}\colon K^*(X) \to K^*(X)$ is the conjugation map.

Consider the Bott sequence

$$\cdots \to K^*(X) \to KO^{*+2}(X) \xrightarrow{\eta} KO^{*+1}(X) \xrightarrow{\mathbf{c}} K^{*+1}(X) \to \cdots,$$

where $\mathbf{c}\colon KO^*(X) \to K^*(X)$ is the complexification map. Since $\mathbf{c}\colon KO^*(X) \to K^*(X)$ factors through $KSC^*(X)$, we have

$$(2)\quad \mathbf{c}\colon KO^*(\mathrm{pt}) \xrightarrow{\cong} KSC^*(\mathrm{pt}) \text{ for } * \equiv 0, -1 \quad (8).$$

3. Proof of Theorem. We denote the n-th stable homotopy group of the sphere by π_n^s and the complex Adams e-invariant of $x \in \pi_{2k-1}^s$ by $e(x)$. By Adams [1] it is shown that:

Lemma 3.1. *For each n there exists $\mu_{8n+1} \in \pi_{8n+1}^s$ such that $e(\mu_{8n+1}) = 1/2$.*

Let X_n be the complex $S^{4m+2} \cup_{\mu_{8n+1}} S^{4m+8n+4}$ for m large enough. Then, by definition of the e-invariant, there exists $\xi \in \tilde{K}^0(X_n)$ such that

$$\mathrm{ch}(\xi) = x_{4m+2} + \frac{1}{2} x_{4m+8n+4},$$

where ch denotes the Chern character and x_k is a generator of $H^k(X_n; \mathbf{Z})$. Therefore we have the following by Bott periodicity of the complex K-theory.

Corollary 3.1.

$$\tilde{K}^{-2l}(X_n) \cong \mathbf{Z}\langle u^l \xi \rangle \oplus \mathbf{Z}\langle u^l(\xi + \mathbf{t}(\xi)) \rangle$$
$$\tilde{K}^{-2l+1}(X_n) = 0,$$

where $u \in K^{-2}(\mathrm{pt})$ is the Bott element.

Since $\mathbf{t} = -1$ on $K^{-2}(\mathrm{pt})$, we have:

Corollary 3.2. *With the basis of $\tilde{K}^{-2l}(X_n)$ in Corollary 3.1, the map $1 - \mathbf{t} : \tilde{K}^{-2l}(X_n) \to$*

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$\tilde{K}^{-2l}(X_n)$ *is represented by* $\left(\begin{smallmatrix} 2 & 0 \\ -1 & 0 \end{smallmatrix}\right)$ *for* l *even and* $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ *for* l *odd.*

By Corollary 3.2 and the exact sequence (1), we have:

Proposition 3.1. $\widetilde{KSC}^*(X_n)$ *is a free abelian group.*

Let $('E_r(X), 'd_r)$ denote the Atiyah-Hirzebruch spectral sequence of $KSC^*(X)$ for a space X. Then we have:

Lemma 3.2. $'d_{8n+2}\colon\ 'E_{8n+2}(X_n) \to\ 'E_{8n+2}(X_n)$ *is non-trivial.*

Proof. Suppose that $'d_{8n+2}\colon\ 'E_r(X_n) \to\ 'E_r(X_n)$ is trivial. Since $'E_2(X_n) \cong\ 'E_\infty(X_n)$ is a free $KSC(\mathrm{pt})$-module, the extension of $'E_\infty(X_n)$ to $KSC^*(X_n)$ is trivial. By (2), $KSC^{-1}(\mathrm{pt}) \cong \mathbf{Z}/2$, then $\widetilde{KSC}^*(X_n)$ has a 2-torsion. This contradicts to Proposition 3.1. □

Finally we prove Theorem 1.1. Since $d_r\colon E_r(X_n) \to E_r(X_n)$ and $'d_r\colon\ 'E_r(X_n) \to\ 'E_r(X_n)$ are trivial for $r < 8n+2$, we have

$$E_{8n+2}(X_n) \cong H^*(X_n; KO^*(\mathrm{pt}))$$
$$'E_{8n+2}(X_n) \cong H^*(X_n; KSC^*(\mathrm{pt})).$$

Consider the homomorphism $\mathbf{c}\colon E_{8n+2}(X_n) \to\ 'E_{8n+2}(X_n)$, then by (2) and Lemma 3.2 we obtain that $d_{8n+2}\colon E_{8n+2}(X_n) \to E_{8n+2}(X_n)$ is non-trivial.

Remark 3.1. At K_*-local, the suspension spectrum of X_n is shown to be the Wood spectrum by Yosimura [6].

References

[1] J.F. Adams, On the groups $J(X)$. IV, Topology **5** (1966), 21–71.

[2] S. Hara and A. Kono, *KO*-theory of complex Grassmannians, J. Math. Kyoto Univ. **31** (1991), no. 3, 827–833.

[3] S. Hara and A. Kono, *KO*-theory of Hermitian symmetric spaces, Hokkaido Math. J. **21** (1992), no. 1, 103–116.

[4] D. Kishimoto, *KO*-theory of complex Stiefel manifolds, J. Math. Kyoto Univ. **44** (2004), no. 3, 669–674.

[5] D. Kishimoto, A. Kono and A. Ohsita, *KO*-theory of flag manifolds, J. Math. Kyoto Univ. **44** (2004), no. 1, 217–227.

[6] Z. Yosimura, The K_*-localizations of Wood and Anderson spectra and the real projective spaces, Osaka J. Math. **29** (1992), no. 2, 361–385.