A comparison theorem on sectors for Kähler magnetic fields

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Abstract: We take sectors as real 2-dimensional objects associated with trajectories for Kähler magnetic fields. As a sequel of [2] we make a bit more consideration on comparison theorem on mangetic Jacobi fields and study lengths of arcs for sectors.

Key words: Sectors; Kähler magnetic fields; trajectories; magnetic Jacobi fields.

1. Introduction. Let $(M, J, \langle\ ,\ \rangle)$ be a Kähler manifold. We call a constant multiple $\mathbf{B}_\kappa = \kappa\mathbf{B}_J$ of the Kähler form $\mathbf{B}_J$ on M a *Kähler magnetic field*. As a generalization of static magnetic fields on a Euclidean 3-space, we generally say a closed 2-form to be a magnetic field. Kähler magnetic fields are typical examples of magnetic fields with uniform strengths. We say a smooth curve γ to be a *trajectory* for $\mathbf{B}_\kappa$ if it satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$. When $\kappa = 0$, trajectories for this trivial magnetic field $\mathbf{B}_0$ are geodesics, therefore we may say that trajectories for Kähler magnetic fields are natural objects from the viewpoint of Riemannian geometry (see [1]).

In Riemannian geometry, it is a natural way to compare the geometry of arbitrary Riemannian manifolds with the geometry of space forms. In the preceding paper [2], in order to study variation of trajectories, we introduced magnetic Jacobi fields and investigated their comparison theorem. Though this corresponds to Rauch's comparison theorem, it is not so powerful because we need to separate a complex direction and totally real directions. In this context we came to study some real 2-dimensional objects associated with Kähler magnetic fields. In [3] we took crescents which are made of trajectories and geodesics and studied their comparison theorem under a condition that sectional curvatures are bounded from above. In this paper, we make a bit more consideration on comparison theorem on magnetic Jacobi fields and investigate how complex lines in a tangent space are mapped through a magnetic exponential map. We take sectors which are images of variations of trajectories and study their comparison theorem under a condition that sectional curvatures are bounded from below.

2. Magnetic Jacobi fields. For a unit tangent vector $u \in TM$ of a Kähler manifold M, we denote by $\gamma_{u,\kappa}$ a trajectory for a Kähler magnetic field $\mathbf{B}_\kappa$ with initial vector u. Given a point $x \in M$ we define the magnetic exponential map $\mathbf{B}_\kappa \exp_x : T_xM \to M$ of the tangent space T_xM at x for $\mathbf{B}_\kappa$ by

$$\mathbf{B}_\kappa \exp_x(v) = \begin{cases} \gamma_{v/\|v\|,\kappa}(\|v\|), & \text{if } v \neq 0_x, \\ x, & \text{if } v = 0_x, \end{cases}$$

where 0_x is the origin of T_xM. For the trivial case that $\kappa = 0$, this is the usual exponential map at x.

Since the differential of magnetic exponential maps corresponds to variations of trajectories, we introduced in [2] the notion of magnetic Jacobi fields. A vector field Y along a trajectory γ for $\mathbf{B}_\kappa$ is said to be a *normal magnetic Jacobi field* for $\mathbf{B}_\kappa$ if it satisfies

i) $\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y - \kappa J\nabla_{\dot\gamma}Y + R(Y, \dot\gamma)\dot\gamma = 0$,

ii) $\nabla_{\dot\gamma}Y \perp \dot\gamma$,

where R denotes the curvature tensor on M. Every normal magnetic Jacobi field is obtained by some variation of trajectories and has similar properties to those of Jacobi fields (see [2] for more detail).

When we study a magnetic Jacobi field, its component orthogonal to a trajectory is important. For a vector field X along a trajectory γ for $\mathbf{B}_\kappa$, we put $X^\sharp = X - \langle X, \dot\gamma\rangle\dot\gamma$. A real number t_0 is said to be a $\mathbf{B}_\kappa$-*conjugate value* for $\gamma(0)$ along γ if there exists a nontrivial normal magnetic Jacobi field Y for $\mathbf{B}_\kappa$ along γ with $Y^\sharp(0) = 0$ and $Y^\sharp(t_0) = 0$. In this case we call $\gamma(t_0)$ a $\mathbf{B}_\kappa$-conjugate point of $\gamma(0)$ along γ. We denote by $t_c(\gamma(0); \gamma, \kappa)$ the minimum positive $\mathbf{B}_\kappa$-conjugate value of $\gamma(0)$ along γ. In our study of Kähler magnetic fields, complex space

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forms, which are complex projective spaces, complex Euclidean spaces and complex hyperbolic spaces, are model spaces. For a complex space form $M^n(c;\mathbf{C})$ of constant holomorphic sectional curvature c, we see this minimum positive $\mathbf{B}_\kappa$-conjugate value is given by

$$t_c(\kappa,c)=\begin{cases}\pi/\sqrt{\kappa^2+c}, & \text{if } \kappa^2+c>0,\\ \infty, & \text{if } \kappa^2+c\leqq 0.\end{cases}$$

More precisely, on $M^n(c;\mathbf{C})$, there are no $\mathbf{B}_\kappa$-conjugate points when $\kappa^2+c\le 0$, and all conjugate values are $\pi j/\sqrt{\kappa^2+c}$, $j=\pm1,\pm2,\cdots$ when $\kappa^2+c>0$.

For a vector field Z along a trajectory γ for $\mathbf{B}_\kappa$ which is perpendicular to $\dot\gamma$, we define its index by

$$\begin{aligned}\mathcal{I}_T(Z)=\int_0^T\bigl\{&(h')^2-\kappa^2h^2\\ &+\langle\nabla_{\dot\gamma}Z^\perp-\kappa JZ^\perp,\nabla_{\dot\gamma}Z^\perp\rangle\\ &-\langle R(Z,\dot\gamma)\dot\gamma,Z\rangle\bigr\}dt,\end{aligned}$$

where $h=\langle Z,J\dot\gamma\rangle$ and $Z^\perp=Z-hJ\dot\gamma$. For a normal magnetic Jacobi field Y along γ, we see

$$\mathcal{I}_T(Y^\sharp)=\langle\nabla_{\dot\gamma}Y^\sharp(T),Y^\sharp(T)\rangle-\langle\nabla_{\dot\gamma}Y^\sharp(0),Y^\sharp(0)\rangle.$$

For a general vector field Z which is perpendicular to $\dot\gamma$ and with initial condition $Z(0)=0$, the following holds: If $Z(T)=Y^\sharp(T)$ for some normal magnetic Jacobi field Y with $Y(0)=0$ at T with $0<T<t_c(\gamma(0);\gamma,\kappa)$, then we see $\mathcal{I}_T(Z)\ge\mathcal{I}_T(Y^\sharp)$, and the equality holds if and only if $Z=Y^\sharp$. Along the lines for the proof of Rauch's comparison theorem, we defined a vector field from a magnetic Jacobi field by parallel transformation and showed the following comparison theorem on magnetic Jacobi fields in [2] by use of the comparison on the index $\mathcal{I}_T$ (see also [4, 5]).

Proposition 1. *Let γ and $\hat\gamma$ be trajectories for Kähler magnetic fields $\mathbf{B}_\kappa$ on Kähler manifolds M and $\hat M$, respectively. Suppose their dimensions satisfy $\dim(M)\ge\dim(\hat M)$, and their sectional curvatures along trajectories satisfy*

$$\min_{v\perp\dot\gamma(t)}\mathrm{Riem}(\dot\gamma(t),v)\ge\max_{\hat v\perp\dot{\hat\gamma}(t)}\mathrm{Riem}(\dot{\hat\gamma}(t),\hat v).$$

We then have the following properties.

1) $t_c(\gamma(0);\gamma,\kappa)\le t_c(\hat\gamma(0);\hat\gamma,\kappa)$.
2) *If normal magnetic Jacobi fields Y and $\hat Y$ along γ and $\hat\gamma$ satisfy $Y^\sharp(0)=\hat Y^\sharp(0)=0$ and $\|\nabla_{\dot\gamma}Y^\sharp(0)\|=\|\nabla_{\dot{\hat\gamma}}\hat Y^\sharp(0)\|$, then for every T with $0\le T<t_c(\gamma(0);\gamma,\kappa)$ we find*

$$\begin{aligned}&\langle\nabla_{\dot\sigma}Y^\sharp(T),Y^\sharp(T)\rangle/\|Y^\sharp(T)\|^2\\ &\qquad\leqq\langle\nabla_{\dot{\hat\sigma}}\hat Y^\sharp(T),\hat Y^\sharp(T)\rangle/\|\hat Y^\sharp(T)\|^2,\end{aligned}\tag{2.1}$$

$$\|Y^\sharp(T)\|\leqq\|\hat Y^\sharp(T)\|.\tag{2.2}$$

If an equality holds in one of these inequalities (2.1) *and* (2.2)*, then for all t with $0\le t\le T$ we see*

i) $\langle\nabla_{\dot\sigma}Y^\sharp(t),Y^\sharp(t)\rangle/\|Y^\sharp(t)\|^2$ $=\langle\nabla_{\dot{\hat\sigma}}\hat Y^\sharp(t),\hat Y^\sharp(t)\rangle/\|\hat Y^\sharp(t)\|^2$,
ii) $\|Y^\sharp(t)\|=\|\hat Y^\sharp(t)\|$,
iii) $\mathrm{Riem}(\dot\gamma(t),Y(t))=\mathrm{Riem}(\dot{\hat\gamma}(t),\hat Y(t))$.

Here we give a bit more consideration on norms of normal magnetic Jacobi fields. If we denote a normal magnetic Jacobi field Y along γ for $\mathbf{B}_\kappa$ as $Y=f\dot\gamma+gJ\dot\gamma+Y^\perp$ with functions f,g and a vector field $Y^\perp$ along γ which is perpendicular to both $\dot\gamma$ and $J\dot\gamma$, the equations for nomal magnetic Jacobi fields turn to

$$f'=\kappa g,\tag{2.3}$$

$$\begin{aligned}&(g''+\kappa^2g)J\dot\gamma+\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y^\perp\\ &\qquad-\kappa J\nabla_{\dot\gamma}Y^\perp+R(Y^\sharp,\dot\gamma)\dot\gamma=0.\end{aligned}\tag{2.4}$$

In order to study images of complex lines through magnetic exponential maps, we need to investigate normal magnetic Jacobi fields whose initial derivatives lie in the complex line spanned by initial vector of a trajectory.

Example 1. On $M^n(c;\mathbf{C})$, as a normal magnetic Jacobi field Y along a trajectory γ for $\mathbf{B}_\kappa$ with initial condition $Y(0)=0$ and $\nabla_{\dot\gamma}Y(0)=J\dot\gamma(0)$ satisfies $Y^\perp\equiv0$, we find its norm $\Lambda(t;\kappa,c)$ is given as follows (c.f. [2]):

(1) When $c\ge0$ or when $c<0$ and $\kappa^2>|c|$, we see

$$\begin{aligned}\Lambda^2(t;\kappa,c)=&\frac{\kappa^2}{(\kappa^2+c)^2}\bigl(1-\cos\sqrt{\kappa^2+c}\,t\bigr)^2\\ &+\frac{1}{\kappa^2+c}\sin^2\sqrt{\kappa^2+c}\,t,\end{aligned}$$

(2) when $c<0$ and $\kappa^2=|c|$, we see

$$\Lambda^2(t;\kappa,c)=(c/4)t^4+t^2,$$

(3) when $c<0$ and $\kappa^2<|c|$, we see

$$\begin{aligned}\Lambda^2(t;\kappa,c)=&\frac{\kappa^2}{(\kappa^2+c)^2}\bigl(\cosh\sqrt{|c|-\kappa^2}\,t-1\bigr)^2\\ &+\frac{1}{\kappa^2+c}\sinh^2\sqrt{|c|-\kappa^2}\,t.\end{aligned}$$

A comparison theorem on norms of magnetic Jacobi fields is given as follows:

Theorem 1. *Let M be a Kähler manifold whose sectional curvature satisfies* $\mathrm{Riem}_M \geq c$. *Then a normal magnetic Jacobi field Y along a trajectory γ for $\mathbf{B}_\kappa$ on M with initial condition $Y(0) = 0$ satisfies*

$$\|Y(T)\| \leq \|\nabla_{\dot\gamma} Y(0)\| \, \Lambda(T;\kappa,c)$$

for $0 \leq T \leq t_c(\gamma(0);\gamma,\kappa)$. The equality holds if and only if $Y^\sharp(t)$ is parallel to $J\dot\gamma(t)$ and the holomorphic sectional curvature $\mathrm{HRiem}(\dot\gamma(t))$ *of the complex line spanned by $\dot\gamma(t)$ is equal to c for all $0 \leq t \leq T$.*

Proof. We take $\hat M$ as a complex space line $M^1(c;\mathbf{C})$. Let $\hat\gamma$ be a trajectory for $\mathbf{B}_{|\kappa|}$ on $M^1(c;\mathbf{C})$, and $\hat Y$ be a normal magnetic Jacobi field along $\hat\gamma$ with initial condition $\hat Y(0) = 0$, $\nabla_{\dot{\hat\gamma}} \hat Y = J\dot{\hat\gamma}(0)$. Since $\hat Y$ does not have a component orthogonal to both $\dot\gamma$ and $J\dot\gamma$, we see by Proposition 1

$$\begin{aligned}\|Y^\sharp(t)\| &\leq \|\nabla_{\dot\gamma} Y(0)\| \, \|\hat Y^\sharp(t)\| \\ &= \|\nabla_{\dot\gamma} Y(0)\| \, \langle \hat Y(t), J\dot{\hat\gamma}(t)\rangle\end{aligned}$$

for $0 \leq t \leq t_c(\gamma(0);\gamma,\kappa)$. By use of (2.3), we find for T with $0 \leq T \leq t_c(\gamma(0);\gamma,\kappa)$ that

$$\begin{aligned}|\langle Y(T),\dot\gamma(T)\rangle| &= |\kappa| \left| \int_0^T \langle Y(t), J\dot\gamma(t)\rangle dt \right| \\ &\leq |\kappa| \int_0^T \|Y^\sharp(t)\| dt \\ &\leq |\kappa| \, \|\nabla_{\dot\gamma} Y(0)\| \int_0^T \langle \hat Y(t), J\dot{\hat\gamma}(t)\rangle dt \\ &= \|\nabla_{\dot\gamma} Y(0)\| \, \langle \hat Y(T), \dot{\hat\gamma}(T)\rangle.\end{aligned} \tag{2.5}$$

Thus we obtain

$$\begin{aligned}\|Y(T)\| &\leq \|\nabla_{\dot\gamma} Y(0)\| \, \|\hat Y^\sharp(T)\| \\ &= \|\nabla_{\dot\gamma} Y(0)\| \, \Lambda(T;\kappa,c).\end{aligned}$$

We now consider the case that the equality holds. If the equality holds, we need by (2.5) that $Y^\perp(t) = 0$ for $0 \leq t \leq T$. Also by Proposition 1 we find

$$\mathrm{HRiem}(\dot\gamma(t)) = \mathrm{Riem}(\dot\gamma(t), Y^\sharp(t)) = c.$$

On the other hand, if Y satisfies $Y^\perp(t) = 0$ and $\mathrm{HRiem}(\dot\gamma(t)) = c$ for $0 \leq t \leq T$, it satisfies the same equation as for a normal magnetic Jacobi field on $M^1(c;\mathbf{C})$. Hence we see the equality holds. □

Example 2. On $M^n(c;\mathbf{C})$ of complex dimension $n \geq 2$, every normal magnetic Jacobi field Y along a trajectory γ for $\mathbf{B}_\kappa$ with initial condition $Y(0) = 0$ and $\langle \nabla_{\dot\gamma} Y(0), J\dot\gamma(0)\rangle = 0$ satisfies $Y = Y^\perp$ and its norm is given by $\|Y(t)\| = \|\nabla_{\dot\gamma} Y(0)\| \lambda(t;\kappa,c)$ for $0 \leq t \leq t_c(\kappa,c)$, where

$$\lambda(t;\kappa,c) = \begin{cases} \dfrac{2}{\sqrt{\kappa^2+c}} \sin \dfrac{\sqrt{\kappa^2+c}}{2}\, t, \\ \qquad \text{when } c \geq 0 \text{ or when } c < 0 \text{ and } \kappa^2 > |c|, \\ t, \quad \text{when } c < 0 \text{ and } \kappa = \pm\sqrt{|c|}, \\ \dfrac{2}{\sqrt{|c|-\kappa^2}} \sinh \dfrac{\sqrt{|c|-\kappa^2}}{2}\, t, \\ \qquad \text{when } c < 0 \text{ and } \kappa^2 < |c|. \end{cases}$$

As an explanation we here give an estimate from below which follows Proposition 1. Let M be a Kähler manifold with $\mathrm{Riem}_M \leq c$. In order to compare normal magnetic Jacobi fields we choose $\mathbf{C}P^n(4c)$ when $c > 0$, $\mathbf{C}^n$ when $c = 0$ and $\mathbf{C}H^n(c)$ when $c < 0$. We then obtain the following as a direct consequence of Proposition 1.

Corollary. *When M is a Kähler manifold whose sectional curvature satisfies* $\mathrm{Riem}_M \leq c$, *then every normal magnetic Jacobi field Y along a trajectory γ for $\mathbf{B}_\kappa$ on M with initial condition $Y(0) = 0$ satisfies*

$$\|Y(T)\| \geq \|\nabla_{\dot\gamma} Y(0)\| \, \lambda(T;\kappa,4c),$$

for $0 \leq T \leq t_c(\kappa,4c)$, when $c > 0$, and

$$\|Y(T)\| \geq \|\nabla_{\dot\gamma} Y(0)\| \, \lambda(T;\kappa,c),$$

for $0 \leq T \leq t_c(\kappa,c)$, when $c \leq 0$.

3. Sectors for Kähler magnetic fields.

In order to study a Kähler manifold M by trajectories for Kähler magnetic fields, it is necessary to take real 2-dimensional objects associated with trajectories. As a candidate of such an object we take images of tangent complex line through magnetic exponential maps. For a unit vector $u \in T_xM$ and real positive numbers r, θ with $0 \leq \theta \leq 2\pi$, we take a tangent sector

$$\mathcal{T}(u,r,\theta) = \left\{ t(\cos s\, u + \sin s\, Ju) \in T_xM \;\middle|\; \begin{matrix} 0 \leq s \leq \theta, \\ 0 \leq t \leq r \end{matrix} \right\}.$$

We call $\mathcal{S}_\kappa(u,r,\theta) = \mathbf{B}_\kappa \exp_x\big(\mathcal{T}(u,r,\theta)\big)$ a $\mathbf{B}_\kappa$-*sector* of radius r and vertical angle θ if $r \leq t_c(x;\gamma_{w_s},\kappa)$ for all s with $0 \leq s \leq \theta$, where $w_s = \cos su + \sin sJu$. For a $\mathbf{B}_\kappa$-sector $\mathcal{S} = \mathcal{S}_\kappa(u,r,\theta)$ we call a curve $\rho_{\mathcal{S}}$ defined by $s \mapsto \mathbf{B}_\kappa \exp_x(rw_s)$ the *arc* of this $\mathbf{B}_\kappa$-sector.

Since every $\mathbf{B}_\kappa$-sector forms a variation of trajectories for $\mathbf{B}_\kappa$, we find the following:

Example 3. On a complex space form $M^n(c;\mathbf{C})$, the length $L(r,\theta;\kappa,c)$ of a $\mathbf{B}_\kappa$-sector is given as

$$L(r,\theta;\kappa,c) = \theta\Lambda(r;\kappa,c).$$

Therefore we find the following:

(1) When $c \geq 0$ or when $c < 0$ and $\kappa^2 > |c|$, we have

$$L(r,\theta;\kappa,c) = \frac{2\theta}{\kappa^2+c}\sin\frac{\sqrt{\kappa^2+c}\,r}{2} \times \sqrt{\kappa^2 + c\cos^2\frac{\sqrt{\kappa^2+c}\,r}{2}},$$

(2) when $c < 0$ and $\kappa^2 = |c|$, we have

$$L(r,\theta;\kappa,c) = \frac{1}{2}\theta r\sqrt{4+|c|r^2},$$

(3) when $c < 0$ and $\kappa^2 < |c|$, we have

$$L(r,\theta;\kappa,c) = \frac{2\theta}{|c|-\kappa^2}\sinh\frac{\sqrt{|c|-\kappa^2}\,r}{2} \times \sqrt{|c|\cosh^2\frac{\sqrt{|c|-\kappa^2}\,r}{2} - \kappa^2}.$$

It is better to explain $\mathbf{B}_\kappa$-sectors on a complex space form in another way. On a complex space form $M^n(c;\mathbf{C})$, every $\mathbf{B}_\kappa$-sector of radius r $\big(\leq t_c(\kappa,c)\big)$ and vertical angle 2π is an intersection of a geodesic ball of radius $\ell(r;\kappa,c)$ and a totally geodesic complex line $M^1(c;\mathbf{C})$. We hence find lengths of $\mathbf{B}_\kappa$-sectors on a complex space form are also given as

$$L(r,\theta;\kappa,c) = \begin{cases} (\theta/\sqrt{c})\sin\sqrt{c}\ell(r;\kappa,c), & \text{when } c > 0, \\ \theta\ell(r;\kappa,0), & \text{when } c = 0, \\ (\theta/\sqrt{|c|})\sinh\sqrt{|c|}\ell(r;\kappa,c), & \text{when } c < 0. \end{cases}$$

Here the distance $\ell(r;\kappa,c)$ between end points of a trajectory segment for $\mathbf{B}_\kappa$ on $M^n(c:\mathbf{C})$ is given by the following relations.

(1) When $c > 0$, it satisfies

$$\sqrt{\kappa^2+c}\sin\big(\sqrt{c}\,\ell(r;\kappa,c)/2\big) = \sqrt{c}\sin\big(\sqrt{\kappa^2+c}\,r/2\big).$$

(2) When $c = 0$, it is given by

$$\ell(r;\kappa,0) = (2/|\kappa|)\sin|\kappa|r/2.$$

(3) When $c < 0$, it satisfies

$$\begin{cases} \sqrt{|c|-\kappa^2}\sinh\dfrac{\sqrt{|c|}\,\ell(r;\kappa,c)}{2} = \sqrt{|c|}\sinh\dfrac{\sqrt{|c|-\kappa^2}\,r}{2}, & \text{if } \kappa^2 < |c|, \\ 2\sinh\big(\sqrt{|c|}\,\ell(r;\kappa,c)/2\big) = \sqrt{|c|}\,r, & \text{if } \kappa^2 = |c|, \\ \sqrt{\kappa^2+c}\sinh\dfrac{\sqrt{|c|}\,\ell(r;\kappa,c)}{2} = \sqrt{|c|}\sin\dfrac{\sqrt{\kappa^2+c}\,r}{2}, & \text{if } \kappa^2 > |c|. \end{cases}$$

We now compare $\mathbf{B}_\kappa$-sectors on a general Kähler manifold with those on a complex space form.

Theorem 2. *Let M be a Kähler manifold whose sectional curvatures satisfy* $\mathrm{Riem}_M \geq c$. *Then for every $\mathbf{B}_\kappa$-sector $\mathcal{S}$ of radius r and vertical angle θ on M, the length* $\mathrm{length}(\rho_{\mathcal{S}})$ *of its arc is not longer than $L(r,\theta;\kappa,c)$. The equality* $\mathrm{length}(\rho_{\mathcal{S}}) = L(r,\theta;\kappa,c)$ *holds if and only if $\mathcal{S}$ is totally geodesic, holomorphic and of constant (holomorphic) sectional curvature c.*

Proof. For a $\mathbf{B}_\kappa$-sector $\mathcal{S} = \mathbf{B}_\kappa\exp_x(\mathcal{T}(u,r,\theta))$, we define a variation of trajectory segments $\alpha : [0,\theta]\times[0,r] \to M$ by $\alpha(s,t) = \mathbf{B}_\kappa\exp_x(tw_s)$, where $w_s = \cos su + \sin sJu$. Then we have

$$\mathrm{length}(\rho_{\mathcal{S}}) = \int_0^\theta \Big\|\frac{\partial\alpha}{\partial s}(s,r)\Big\| ds.$$

As $\partial\alpha/\partial s(s,\cdot)$ is a normal magnetic Jacobi field along a $\mathbf{B}_\kappa$-trajectory $\alpha(s,\cdot)$ and satisfies

$$\begin{cases} \dfrac{\partial\alpha}{\partial s}(s,0) = 0, \\ \nabla_{\frac{\partial\alpha}{\partial t}}\dfrac{\partial\alpha}{\partial s}(s,0) = -\sin s\,u + \cos s\,Ju, \end{cases}$$

we obtain our conclusion by Theorem 1. □

By standing another point of view, we see Theorem 2 assures the following:

Corollary. *If a $\mathbf{B}_\kappa$-sector $\mathcal{S}$ of radius r on a Kähler manifold M with* $\mathrm{Riem}_M \geq c$ *has an arc of length $L(r,\theta;\kappa,c)$, then its vertical angle is not less than θ. The vertical angle is equal to θ if and only if $\mathcal{S}$ is totally geodesic, holomorphic and of constant (holomorphic) sectional curvature c.*

Unfortunately our proof does not use essentially the property that our $\mathbf{B}_\kappa$-sectors are obtained as images of subsets of complex lines. As a matter of course, if we take

$$\mathcal{T}(u, v; t, \theta) = \left\{ t(\cos s\, u + \sin s\, v) \in T_xM \;\middle|\; \begin{matrix} 0 \le s \le \theta, \\ 0 \le t \le r \end{matrix} \right\}$$

for an orthonormal pair $(u, v) \in T_xM \times T_xM$ of unit tangent vectors and put

$$\mathcal{S}_\kappa(u, v; r, \theta) = \mathbf{B}_\kappa \exp_x\bigl(\mathcal{T}(u, v; r, \theta)\bigr),$$

we see the length of its arc satisfies the same estimate as in Theorem 2 under the same condition. This is because we pose an assumption on sectional curvatures. On $M^n(c;\mathbf{C})$, if $(u, v) \in TM^n(c;\mathbf{C}) \times TM^n(c;\mathbf{C})$ is a totally real orthonormal pair, which means they satisfy $\langle u, Jv\rangle = 0$, we find the length of the arc of $\mathcal{S}_\kappa(u, v; r, \theta)$ is $\theta\lambda(r; \kappa, c)$ when $r \le t_c(\kappa, c)$. Thus we can obtain a trivial estimate of lengths of arcs from below under a condition that sectional curvatures are bounded from above by use of Corollary.

We should note that trajectories for Kähler magnetic fields are also called *Kähler circles*. In submanifold theory some results are obtained by use of some properties of Kähler circles (see [6] and its references).

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