Secondary Whittaker functions for P_J-principal series representations of $Sp(3, \mathrm{R})$

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Abstract: In this paper, we give explicit formulas for the secondary Whittaker functions for P_J-principal series representations of $Sp(3, \mathbf{R})$, which are power series solutions of a holonomic system of rank 24.

Key words: Whittaker functions; Whittaker models.

1. Introduction. We discuss in this paper the secondary Whittaker functions for a generalized principal series representation of $Sp(3, \mathbf{R})$. Here the secondary functions mean the power series solutions at the regular singularity of the holonomic system coming from the realization of the representation, which is originated in Harish-Chandra's study [2] on the matrix coefficients. In view of automorphic forms, it is known that these are fundamental in constructing the Poincaré series (*cf.* [9, 10]). Moreover, the functions obtained in this paper give concrete examples of confluent hypergeometric series of three variables, which are not simple Γ-series. It is also interesting to compare this result with the other ones (*cf.* [3, 5, 13]).

2. Preliminaries.

2.1. Groups and algebras. Let $M_n(\mathbf{R})$ be the space of real matrices of size n and 1_n (resp. O_n) be the unit (resp. the zero) matrix in $M_n(\mathbf{R})$. The real symplectic group $G = Sp(3, \mathbf{R})$ of degree three is defined by

$$G = Sp(3, \mathbf{R}) = \left\{ g \in M_6(\mathbf{R}) \mid {}^t g J_3 = J_3 g^{-1},\ \det g = 1 \right\},$$

with $J_3 = \begin{pmatrix} O_3 & 1_3 \\ -1_3 & O_3 \end{pmatrix}$, which is connected, semisimple, and split over $\mathbf{R}$. Let $\theta(g) = {}^t g^{-1}$, $g \in G$, be a Cartan involution of G. Then $K = \{g \in G \mid \theta(g) = g\}$ is a maximal compact subgroup of G which is isomorphic to the unitary group $U(3)$ of degree three.

Let $\mathfrak{g} = \mathfrak{sp}(3, \mathbf{R})$ be the Lie algebra of G and $\mathfrak{k}$ (resp. $\mathfrak{p}$) be the $+1$ (resp. -1) eigenspace of the differential of θ in $\mathfrak{g}$. Then $\mathfrak{k}$ is the Lie algebra of K which is isomorphic to the unitary algebra $\mathfrak{u}(3)$ of degree three and $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Now we fix an isomorphism κ between $\mathfrak{u}(3)$ and $\mathfrak{k}$ given by

$$\kappa : \mathfrak{u}(3) \ni X \mapsto \frac{1}{2} \begin{pmatrix} X + \bar{X} & \sqrt{-1}(\bar{X} - X) \\ \sqrt{-1}(X - \bar{X}) & X + \bar{X} \end{pmatrix} \in \mathfrak{k}.$$

For a Lie algebra $\mathfrak{l}$, we denote by $\mathfrak{l}_{\mathbf{C}} = \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of $\mathfrak{l}$. Take a compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ defined by $\mathfrak{h} = \bigoplus_{i=1}^{3} \mathbf{R} T_i$ with $T_i = \kappa(\sqrt{-1}E_{ii}) \in \mathfrak{k}$, where E_{ij} is the matrix unit in $M_3(\mathbf{R})$ with (i, j) entry. For each $1 \le i \le 3$, define a linear form β_i on $\mathfrak{h}_{\mathbf{C}}$ by $\beta_i(T_j) = \sqrt{-1}\delta_{ij}$, $1 \le j \le 3$. Here δ_{ij} is the Kronecker's delta. Then the set Δ of roots of $(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ is given by $\Delta = \Delta(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}}) = \{\pm 2\beta_i,\ \pm\beta_j \pm \beta_k\ (j < k)\}$ and the subset $\Delta^+ = \{2\beta_i,\ \beta_j \pm \beta_k\ (j < k)\}$ forms a positive root system. Let $\Delta_c^+ = \{\beta_i - \beta_j\ (i < j)\}$ and $\Delta_n^+ = \{2\beta_i,\ \beta_j + \beta_k\ (j < k)\}$ be the set of compact and non-compact positive roots, respectively. If we denote the root space for $\beta \in \Delta$ by $\mathfrak{g}_\beta$, then $\mathfrak{k}_{\mathbf{C}} \simeq \mathfrak{gl}(3, \mathbf{C})$ and $\mathfrak{p}_{\mathbf{C}}$ have the decompositions

$$\mathfrak{k}_{\mathbf{C}} = \mathfrak{h}_{\mathbf{C}} \oplus \left(\bigoplus\nolimits_{\beta \in \Delta_c^+} \mathfrak{g}_{\pm\beta} \right),$$

$$\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-, \quad \mathfrak{p}_\pm = \bigoplus\nolimits_{\beta \in \Delta_n^+} \mathfrak{g}_{\pm\beta}.$$

Now we take a basis of $\mathfrak{k}_{\mathbf{C}}$ and $\mathfrak{p}_\pm$ consisting of root vectors. Let us denote the extension of the isomorphism κ to their complexifications again by κ. Then we have $\kappa(E_{ij}) \in \mathfrak{g}_{\beta_i - \beta_j}$ for $i \ne j$ and thus the set $\{\kappa(E_{ij}) \mid 1 \le i, j \le 3\}$ forms a basis of $\mathfrak{k}_{\mathbf{C}}$. Define a

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$$p_\pm : \{X \in M_3(\mathbf{C}) \mid X = {}^tX\} \ni X \mapsto \begin{pmatrix} X & \pm\sqrt{-1}X \\ \pm\sqrt{-1}X & -X \end{pmatrix} \in \mathfrak{p}_\pm.$$

Then the element $X_{\pm ij} = p_\pm\left(\frac{1}{2}(E_{ij}+E_{ji})\right)$ is a root vector in $\mathfrak{g}_{\pm(\beta_i+\beta_j)}$ for $i \le j$ and the set $\{X_{\pm ij} \mid 1 \le i \le j \le 3\}$ gives a basis of $\mathfrak{p}_\pm$.

Take a maximal abelian subalgebra $\mathfrak{a}_\mathfrak{p} = \bigoplus_{i=1}^3 \mathbf{R}H_i$ of $\mathfrak{p}$ with $H_1 = \mathrm{diag}(1,0,0,-1,0,0)$, $H_2 = \mathrm{diag}(0,1,0,0,-1,0)$, and $H_3 = \mathrm{diag}(0,0,1,0,0,-1)$, and define $e_i \in \mathfrak{a}_\mathfrak{p}^*$ for each $1 \le i \le 3$ by $e_i(H_j) = \delta_{ij}$ for $1 \le j \le 3$. Then the set Σ of the restricted roots of $(\mathfrak{a}_\mathfrak{p}, \mathfrak{g})$ is given by $\Sigma = \Sigma(\mathfrak{a}_\mathfrak{p}, \mathfrak{g}) = \{\pm 2e_i,\ \pm e_j \pm e_k\ (j<k)\}$ and the subset $\Sigma^+ = \{2e_i,\ e_j \pm e_k\ (j<k)\}$ forms a positive root system. For each $\alpha \in \Sigma$, we denote the restricted root space by $\mathfrak{g}_\alpha$ and choose a restricted root vector E_α in $\mathfrak{g}_\alpha$. If we put $\mathfrak{n}_\mathfrak{p} = \bigoplus_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$, then $\mathfrak{g}$ has an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{p} \oplus \mathfrak{k}$. Also we have $G = NAK$, where A (resp. N) is the analytic subgroup with Lie algebra $\mathfrak{a}_\mathfrak{p}$ (resp. $\mathfrak{n}_\mathfrak{p}$).

Set

$$\mathfrak{a}_J = \bigoplus_{i=1}^2 \mathbf{R}H_i, \quad \mathfrak{n}_J = \bigoplus_{\alpha\in\Sigma^+\setminus\{2e_3\}} \mathfrak{g}_\alpha,$$
$$\mathfrak{m}_J = \mathbf{R}H_3 \oplus \mathfrak{g}_{2e_3} \oplus \mathfrak{g}_{-2e_3}.$$

Moreover let A_J, N_J, $M_{J,0}$ be the analytic subroups with Lie algebras $\mathfrak{a}_J$, $\mathfrak{n}_J$, $\mathfrak{m}_J$, respectively. Then $P_J = M_JA_JN_J$ with $M_J = Z_K(\mathfrak{a}_J)M_{J,0}$ is a parabolic subgroup of G corresponding to the root $2e_3$ and the right-hand side gives its Langlands decomposition. Here $Z_K(\mathfrak{a}_J) = \{1_6, \mu_1\} \times \{1_6, \mu_2\}$ with $\mu_i = \exp \pi T_i$ is the centralizer of $\mathfrak{a}_J$ in K. We call P_J *the second Jacobi parabolic subgroup* of G.

2.2. Representations. Here we introduce some notations for representations of K, G, and N which we need in this paper.

The equivalence classes of irreducible representations of $K \simeq U(3)$ can be parameterized by the set $\Lambda = \{\lambda = (\lambda_1, \lambda_2, \lambda_3) \mid \lambda_i \in \mathbf{Z}, \lambda_1 \ge \lambda_2 \ge \lambda_3\}$. If we denote the representation of K associated to $\lambda \in \Lambda$ by $(\tau_\lambda, V_\lambda)$, the representation space V_λ has the Gelfand-Zelevinsky basis $\{f(M)\}_{M\in G(\lambda)}$ parameterized by the set $G(\lambda)$ of all G-patterns of type λ (*cf.* [1, 4]). Here a G-pattern $M \in G(\lambda)$ of type $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda$ is a triangular array

$$M = \begin{pmatrix} \lambda_1\ \lambda_2\ \lambda_3 \\ \alpha_1\ \alpha_2 \\ \beta \end{pmatrix}$$

of integers satisfying the conditions $\lambda_1 \ge \alpha_1 \ge \lambda_2 \ge \alpha_2 \ge \lambda_3$ and $\alpha_1 \ge \beta \ge \alpha_2$. When $\lambda = (m,m,m) \in \Lambda$, the associated representation $(\tau_\lambda, V_\lambda)$ is one dimensional and

$$\tau_\lambda(\kappa(E_{ii}))v = mv, \quad (1 \le i \le 3),$$
$$\tau_\lambda(\kappa(E_{ij}))v = 0, \quad (i \ne j), \quad v \in V_\lambda.$$

Moreover, it is known that both of $\mathfrak{p}_\pm$ become K-modules via the adjoint action of K, and we have isomorphisms $\mathfrak{p}_+ \simeq V_{(2,0,0)}$ and $\mathfrak{p}_- \simeq V_{(0,0,-2)}$.

Let $\sigma = (\varepsilon_1, \varepsilon_2, D)$ be a representation of M_J with characters $\varepsilon_i : \{1_6, \mu_i\} \to \mathbf{C}^\times$ and a (limit of) discrete series representation $D = \mathcal{D}_k^\pm$ of $M_{J,0} \simeq SL(2,\mathbf{R})$ with the Blattner parameter $\pm k$ ($k \in \mathbf{Z}_{\ge 1}$), and take a quasi-character ν of A_J. Then we can construct an induced representation $\mathrm{Ind}_{P_J}^G(\sigma \otimes \nu \otimes 1_{N_J})$ of G from the second Jacobi parabolic subgroup P_J in the usual manner, which we call *a P_J-principal series representation* of G. The multiplicity theorem for the K-types can be computed by the Frobenius reciprocity for induced representations.

Proposition 2.1. *Put* $\mathrm{sgn}(D) = 1$ (*resp.* -1) *for* $D = \mathcal{D}_k^+$ (*resp.* $\mathcal{D}_k^-$). *Then each irreducible K-module $(\tau_\lambda, V_\lambda)$ with $\lambda \in \Lambda$ occurs in the restriction of $\mathrm{Ind}_{P_J}^G(\sigma \otimes \nu \otimes 1_{N_J})$ to K with the following multiplicity m_λ.*

$$m_\lambda = \#\left\{ M \in G(\lambda) \;\middle|\; \begin{matrix} \varepsilon_i(\mu_i) = (-1)^{w_i}, \ \ i=1,2 \\ k \equiv w_3 \pmod 2 \\ k \le \mathrm{sgn}(D) w_3 \end{matrix} \right\}.$$

Here $w = (w_1, w_2, w_3)$ is the weight for $M \in G(\lambda)$ defined by the formula

$$w_1 = \beta, \quad w_2 = \alpha_1 + \alpha_2 - \beta, \quad w_3 = \lambda_1 + \lambda_2 + \lambda_3 - \alpha_1 - \alpha_2.$$

For $\pi = \mathrm{Ind}_{P_J}^G(\sigma \otimes \nu \otimes 1_{N_J})$ with $\sigma = (\varepsilon_1, \varepsilon_2, \mathcal{D}_k^+)$ such that $\varepsilon_i(\mu_i) = (-1)^k$, one can see from the above formula that $m_{(k,k,k)} = 1$ and $m_{(k-2,k-2,k-2)} = 0$. The K-type $\tau_{(k,k,k)}$ of π is called *corner*.

Let η be a unitary character of N and denote the derivative of η by the same letter. Since $\mathfrak{n}_\mathfrak{p}^{\mathrm{ab}} = \mathfrak{n}_\mathfrak{p}/[\mathfrak{n}_\mathfrak{p}, \mathfrak{n}_\mathfrak{p}] \simeq \mathfrak{g}_{e_1-e_2} \oplus \mathfrak{g}_{e_2-e_3} \oplus \mathfrak{g}_{2e_3}$, η is specified by three real numbers c_{12}, c_{23}, and c_3 such that

$$\eta(E_{e_1-e_2}) = 2\pi\sqrt{-1}c_{12}, \quad \eta(E_{e_2-e_3}) = 2\pi\sqrt{-1}c_{23},$$
$$\eta(E_{2e_3}) = 2\pi\sqrt{-1}c_3.$$

When $c_{12}c_{23}c_3 \neq 0$, a unitary character η of N is called *non-degenerate.*

3. Whittaker functions. For a finite dimensional representation (τ, V_τ) of K and a non-degenerate unitary character η of N, we consider the space $C^\infty_{\eta,\tau}(N\backslash G/K)$ of smooth functions $\varphi : G \to V_\tau$ with the property

$$\varphi(ngk) = \eta(n)\tau(k)^{-1}\varphi(g), \quad (n, g, k) \in N \times G \times K.$$

Here we remark that any function $f \in C^\infty_{\eta,\tau}(N\backslash G/K)$ is determined by its restriction $f|_A$ to A from the Iwasawa decomposition $G = NAK$ of G. Let (τ^*, V_{τ^*}) be the contragradient representation of (τ, V_τ) and $C^\infty \operatorname{Ind}_N^G(\eta)$ be the C^∞-induced representation from η. Then the space $C^\infty_{\eta,\tau}(N\backslash G/K)$ is isomorphic to $\operatorname{Hom}_K(\tau^*, C^\infty \operatorname{Ind}_N^G(\eta))$ via the correspondence between $\iota \in \operatorname{Hom}_K(\tau^*, C^\infty \operatorname{Ind}_N^G(\eta))$ and $F^{[\iota]} \in C^\infty_{\eta,\tau}(N\backslash G/K)$ given by the relation $\iota(v^*)(g) = \langle v^*, F^{[\iota]}(g)\rangle$ for $v^* \in V_{\tau^*}$ and $g \in G$ with the canonical bilinear form $\langle\cdot,\cdot\rangle$ on $V_{\tau^*} \times V_\tau$.

Let (π, H_π) be an irreducible admissible representation of G, and take a multiplicity one K-type (τ^*, V_{τ^*}) of π with an injection $i : \tau^* \to \pi$. Then, for each element T in the intertwining space $\mathcal{I}_{\eta,\pi} = \operatorname{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(\pi, C^\infty \operatorname{Ind}_N^G(\eta))$ between $(\mathfrak{g}_{\mathbf{C}}, K)$-modules consisting of all K-finite vectors, the relation $T(i(v^*))(g) = \langle v^*, T_i(g)\rangle$ for $v^* \in V_{\tau^*}$ and $g \in G$ determines an element $T_i \in C^\infty_{\eta,\tau}(N\backslash G/K)$. Now we put

$$\begin{aligned}&\operatorname{Wh}(\pi, \eta, \tau)\\ &\quad = \bigcup_{i\in \operatorname{Hom}_K(\tau^*,\pi)} \{T_i \in C^\infty_{\eta,\tau}(N\backslash G/K) \mid T \in \mathcal{I}_{\eta,\pi}\},\end{aligned}$$

and call $\operatorname{Wh}(\pi, \eta, \tau)$ *the space of Whittaker functions* for (π, η, τ).

4. Differential equations. Let $\pi = \operatorname{Ind}_{P_J}^G(\sigma \otimes \nu \otimes 1_{N_J})$ be an irreducible P_J-principal series representation of G with $\sigma = (\varepsilon_1, \varepsilon_2, \mathcal{D}_k^+)$ and ν such that $\varepsilon_i(\mu_i) = (-1)^k$ and $\nu(\operatorname{diag}(a_1, a_2, 1, a_1^{-1}, a_2^{-1}, 1)) = a_1^{\nu_1} a_2^{\nu_2}$, and let τ^* be the corner K-type of π, that is, $\tau = \tau_{(-k,-k,-k)}$. Moreover, let η be a non-degenerate unitary character of N specified by three real numbers c_{12}, c_{23} and c_3. In the rest of this paper, we discuss the space $\operatorname{Wh}(\pi, \eta, \tau)$ of Whittaker functions for the above (π, η, τ).

Definition 4.1. The $\pm$-chirality matrices $m_i(C_\pm)$ for $1 \le i \le 3$ are defined by

$$m_1(C_\pm) = \begin{bmatrix} X_{\pm 11} & X_{\pm 12} & X_{\pm 13} \\ X_{\pm 12} & X_{\pm 22} & X_{\pm 23} \\ X_{\pm 13} & X_{\pm 23} & X_{\pm 33}\end{bmatrix},$$

$$m_2(C_\pm) = \begin{bmatrix} M_{\pm 11} & -M_{\pm 12} & M_{\pm 13} \\ -M_{\pm 12} & M_{\pm 22} & -M_{\pm 23} \\ M_{\pm 13} & -M_{\pm 23} & M_{\pm 33}\end{bmatrix},$$

and $m_3(C_\pm) = \det(m_1(C_\pm))$. Here $M_{\pm ij}$ is the (i, j)-minor of the matrix $m_1(C_\pm)$ for each $1 \le i \le j \le 3$.

Put $C_{2i} = \operatorname{Tr}(m_i(C_+)m_i(C_-))$ for $1 \le i \le 3$. Then one can see that $C_{2i} \in U(\mathfrak{g}_{\mathbf{C}})^K = \{X \in U(\mathfrak{g}_{\mathbf{C}}) \mid \operatorname{Ad}(k)X = X, k \in K\}$.

Remark 4.2. In the case of $Sp(n, \mathbf{R})$, we can define C_{2i} for $1 \le i \le n$ belonging to $U(\mathfrak{g}_{\mathbf{C}})^K$ similarly. The operator C_{2n} is essentially the same as the so-called Maass shift operator in the classical literature [7].

The explicit actions of the operators C_{2i} on $C^\infty_{\eta,\tau}(N\backslash G/K)$ can be computed by expressing them in the normal order modulo $[\mathfrak{n}_{\mathfrak{p}}, \mathfrak{n}_{\mathfrak{p}}]$ with respect to the Iwasawa decomposition of $\mathfrak{g}$, according to the following fundamental lemma.

Lemma 4.3. *Let $f \in C^\infty_{\eta,\tau}(N\backslash G/K)$. For $X \in U(\mathfrak{k}_{\mathbf{C}})$, $Y \in U(\mathfrak{n}_{\mathfrak{p}\mathbf{C}})$, $Z \in U(\mathfrak{a}_{\mathfrak{p}\mathbf{C}})$ and $a \in A$, we have $(\operatorname{Ad}(a^{-1})Y)ZXf(a) = \eta(Y)\tau(-X)(Zf)(a)$. In particular, for $a = \operatorname{diag}(a_1, a_2, a_3, a_1^{-1}, a_2^{-1}, a_3^{-1}) \in A$, we have $H_i f(a) = a_i \frac{\partial}{\partial a_i} f(a)$ and*

$$E_{e_1-e_2}f(a) = 2\pi\sqrt{-1}c_{12}\frac{a_1}{a_2}f(a),$$
$$E_{e_2-e_3}f(a) = 2\pi\sqrt{-1}c_{23}\frac{a_2}{a_3}f(a),$$
$$E_{2e_3}f(a) = 2\pi\sqrt{-1}c_3 a_3^2 f(a),$$

and $E_\alpha f(a) = 0$ for $\forall\alpha \in \Sigma^+\backslash\{e_1 - e_2, e_2 - e_3, 2e_3\}$.

Now we consider a holonomic system of partial differential equations which is satisfied by the A-radial part of each element in $\operatorname{Wh}(\pi, \eta, \tau)$. For our convenience, we use a coordinate $x = (x_1, x_2, x_3)$ on A with $x_1 = \left(\pi c_{12}\frac{a_1}{a_2}\right)^2$, $x_2 = \left(\pi c_{23}\frac{a_2}{a_3}\right)^2$ and $x_3 = 4\pi c_3 a_3^2$ for $\operatorname{diag}(a_1, a_2, a_3, a_1^{-1}, a_2^{-1}, a_3^{-1}) \in A$.

Theorem 4.4. *Each element φ in the space $\operatorname{Wh}(\pi, \eta, \tau)|_A$ of the restriction of Whittaker functions to A satisfies the following holonomic system of partial differential equations of rank 24:*

$$(1) \qquad \begin{cases} \mathcal{D}_2\varphi(x) = 0, \\ \mathcal{D}_3\varphi(x) = 0, \\ \mathcal{D}_4\varphi(x) = 0. \end{cases}$$

Here

$$
\begin{aligned}
\mathcal{D}_2 &= (2\partial_1 + k - 6)(2\partial_1 - k) \\
&\quad + (-2\partial_1 + 2\partial_2 + k - 4)(-2\partial_1 + 2\partial_2 - k) \\
&\quad + (-2\partial_2 + 2\partial_3 - x_3 + k - 2) \\
&\qquad \cdot (-2\partial_2 + 2\partial_3 + x_3 - k) \\
&\quad - 8x_1 - 8x_2 - \chi_{2,k,\nu}, \\
\mathcal{D}_3 &= (2\partial_1 - k - 2) \\
&\qquad \cdot \{(-2\partial_1 + 2\partial_2 - k - 1) \\
&\qquad\quad \cdot (-2\partial_2 + 2\partial_3 + x_3 - k) + 4x_2\} \\
&\quad + 4x_1(-2\partial_2 + 2\partial_3 + x_3 - k), \\
\mathcal{D}_4 &= \{(-2\partial_1 + 2\partial_2 + k - 3) \\
&\qquad\quad \cdot (-2\partial_2 + 2\partial_3 - x_3 + k - 2) + 4x_2\} \\
&\qquad \cdot \{(-2\partial_1 + 2\partial_2 - k - 1) \\
&\qquad\quad \cdot (-2\partial_2 + 2\partial_3 + x_3 - k) + 4x_2\} \\
&\quad + (2\partial_1 + k - 5)(-2\partial_2 + 2\partial_3 - x_3 + k - 2) \\
&\qquad \cdot (2\partial_1 - k - 1)(-2\partial_2 + 2\partial_3 + x_3 - k) \\
&\quad + \{(2\partial_1 + k - 5) \\
&\qquad\quad \cdot (-2\partial_1 + 2\partial_2 + k - 4) + 4x_1\} \\
&\qquad \cdot \{(2\partial_1 - k - 1)(-2\partial_1 + 2\partial_2 - k) + 4x_1\} \\
&\quad - 8x_1(-2\partial_2 + 2\partial_3 - x_3 + k - 2) \\
&\qquad \cdot (-2\partial_2 + 2\partial_3 + x_3 - k) \\
&\quad + 32x_1x_2 - 8x_2(2\partial_1 + k - 5)(2\partial_1 - k - 1) \\
&\quad - \chi_{4,k,\nu}, \\
\chi_{2,k,\nu} &= \{\nu_1^2 - (k-3)^2\} + \{\nu_2^2 - (k-2)^2\}, \\
\chi_{4,k,\nu} &= \{\nu_1^2 - (k-2)^2\}\{\nu_2^2 - (k-2)^2\},
\end{aligned}
$$

and $\partial_i = x_i \frac{\partial}{\partial x_i}$ *is the Euler operator with respect to the variable* x_i.

Proof. It is easy to see that the eigenvalues of the scalar actions of C_2 and C_4 on $\mathrm{Wh}(\pi, \eta, \tau)$ are $\chi_{2,k,\nu}$ and $\chi_{4,k,\nu}$, respectively. By computing their explicit actions from Lemma 4.3, the equations $\mathcal{D}_2\varphi = \mathcal{D}_4\varphi = 0$ can be obtained. The operator $m_3(C_-)$ maps the K-type $\tau^* = \tau_{(k,k,k)}$ into $\tau_{(k-2,k-2,k-2)}$ in the Harish-Chandra module of π from the definition. However, since $\tau_{(k,k,k)}$ is the corner K-type of π, the K-module $\tau_{(k-2,k-2,k-2)}$ does not occur in the K-types of π. Therefore each element in $\mathrm{Wh}(\pi, \eta, \tau)$ vanishes by the action of $m_3(C_-)$, and the equation $\mathcal{D}_3\varphi = 0$ follows. □

By combining the results of Kostant ([6] Theorem 6.8.1) and Matumoto ([8] Corollary 2.2.2, Theorem 6.2.1) together with some standard arguments, we obtain the following assertion.

Corollary 4.5. *Let* π, τ, *and* η *be as above. Then we have*

$$\dim_{\mathbf{C}} \mathcal{I}_{\eta,\pi} = \dim_{\mathbf{C}} \mathrm{Wh}(\pi, \eta, \tau) = \frac{1}{2}|W| = 24,$$

and thus every solution of the holonomic system in Theorem 4.4 *gives an element of* $\mathrm{Wh}(\pi, \eta, \tau)|_A$. *Here* $W \simeq \{\pm 1\}^3 \times \mathfrak{S}_3$ *is the little Weyl group* $W(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$.

5. Secondary Whittaker functions. The holonomic system (1) has regular singularities along 3 divisors $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ with normal crossing at $x = (0, 0, 0)$, in the sense of [11]. In this section, we determine the power series solutions of the system (1) around the point $x = (0, 0, 0)$, which are called the secondary Whittaker functions.

First, we give 24 characteristic indices for the secondary Whittaker functions. For a characteristic index $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ for the holonomic system (1), put $\delta = (\delta_1, \delta_2, \delta_3) = (\gamma_1, -\gamma_1 + \gamma_2, -\gamma_2 + \gamma_3)$. Then we have

$$\delta = \sigma\left(\frac{\epsilon_1\nu_1}{2}, \frac{\epsilon_2\nu_2}{2}, \frac{k-1}{2}\right), \tag{2}$$

with $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ and $\sigma \in \mathfrak{S}_3$. Here $\mathfrak{S}_3$ means the symmetric group of degree 3.

The explicit secondary Whittaker functions for (π, η, τ) are given in the following theorem, which is the main result in this paper.

Theorem 5.1. *For each* $\gamma \in \mathbf{C}^3$ *such that* δ *is given in* (2), *put*

$$
\begin{aligned}
M_\gamma(x) &= x_1^{3/2+\gamma_1} x_2^{5/2+\gamma_2} x_3^{3+\gamma_3} \exp\left(-\frac{x_3}{2}\right) \\
&\quad \times \sum_{l,m,n \ge 0} C^\gamma_{l,m,n} x_1^l x_2^m x_3^n,
\end{aligned}
$$

where the coefficients $\{C^\gamma_{l,m,n}\}$ *are defined as follows: For* $l, m, n \in \mathbf{Z}_{\ge 0}$ *and constants* a, b, c, a', b', c', *put*

$$
\begin{aligned}
k_{l,m,n} &= k_{l,m,n}(a, b, c, a', b', c') \\
&= \frac{1}{n!} \cdot \frac{(m+a)_n(-l+b)_n}{(c)_n} \\
&\quad \times {}_4F_3\left(\left.\begin{matrix} -n,\ 1-n-c,\ -m+a',\ l+b' \\ 1-n-m-a,\ 1-n+l-b,\ c' \end{matrix}\right| 1\right),
\end{aligned}
$$

where $(a)_n$ *is Pochhammer's symbol and* ${}_pF_q$ *is the generalized hypergeometric function* (*cf.* [12]). *If either* δ_1 *or* δ_2 *is* $\frac{k-1}{2}$, *then*

$$
\begin{aligned}
C^\gamma_{l,m,n} &= \frac{1}{m!}\Gamma\left[\begin{matrix} l+m-n+\alpha_2,\ \alpha_1,\ \alpha_3 \\ m-n+\alpha_1,\ l+\alpha_2,\ m+\alpha_3 \end{matrix}\right] \\
&\quad \times \Gamma\left[\begin{matrix} \alpha_4,\ \alpha_5,\ \alpha_6 \\ n+\alpha_4,\ l+\alpha_5,\ l+\alpha_6 \end{matrix}\right]
\end{aligned}
$$

$$\times k_{l,m,n}(\alpha_4, -\alpha_2+1, -\alpha_3+\alpha_4+1, 0, \alpha_2+\alpha_4-1, \alpha_3+\alpha_4-1),$$

with

$$\alpha_1 = -\delta_3 + \frac{k+1}{2}, \quad \alpha_2 = \delta_1 - \delta_3 + 1,$$
$$\alpha_3 = \delta_* - \delta_3 + 1, \quad \alpha_4 = \delta_* + \delta_3 + 1,$$
$$\alpha_5 = \delta_1 - \frac{k-3}{2}, \quad \alpha_6 = -\delta_2 + \frac{k+1}{2},$$

and $\delta_* = \delta_1 + \delta_2 - \frac{k-1}{2}$. *If* $\delta_3 = \frac{k-1}{2}$, *then*

$$C^{\gamma}_{l,m,n} = \frac{1}{(m-n)!l!}\Gamma\begin{bmatrix} l+m-n+\beta_1,\ \beta_2 \\ l+\beta_1,\ l+\beta_2 \end{bmatrix} \times \Gamma\begin{bmatrix} \beta_3,\ \beta_1,\ \beta_4 \\ n+\beta_3,\ m+\beta_1,\ m+\beta_4 \end{bmatrix} \times k_{m,l,n}(\beta_3,\ -\beta_1+1,\ -\beta_2+\beta_3+1,\ 0,\ \beta_1+\beta_3-1,\ \beta_2+\beta_3-1),$$

for $m \geq n$ *and* $C^{\gamma}_{l,m,n} = 0$ *for* $m < n$, *where*

$$\beta_1 = \delta_1 - \frac{k-3}{2}, \quad \beta_2 = \delta_1 - \delta_2 + 1,$$
$$\beta_3 = \delta_1 + \delta_2 + 1, \quad \beta_4 = \delta_2 - \frac{k-3}{2}.$$

Then, the set $\{M_\gamma(x)\}$ *gives a system of linearly independent solutions of the holonomic system* (1) *at* $x = (0,0,0)$.

Proof. To obtain this result, we transform the system (1) into a system of difference equations for the coefficients of formal power series solutions. The resulted system can be reduced the difference equations in the next lemma. □

Lemma 5.2. *For any constants* a, b, c, a', b', c' *such that* $c, c' \notin \mathbf{Z}_{\leq 0}$ *and* $a, b \notin \mathbf{Z}$, $\{k_{l,m,n}\}$ *satisfies the following two difference equations.*

$$f_1(l,m,n)k_{l,m,n} = f_2(l,m,n)k_{l,m,n-1} + 2(m-a')(m+a-c)k_{l,m-1,n},$$
$$g(l,m,n)k_{l,m,n} = (m-a')(m+a-c)k_{l,m-1,n} - (l-b)(l+b'-c')k_{l-1,m,n}.$$

Here

$$f_1(l,m) = n^2 + (-2m+2a'+c-1)n + 2(m-a')(m+a-c),$$
$$f_2(l,m,n) = n^2 - (m+l-2a'-a-b+2)n - (a+a'-1)(m+l-a'-b+1),$$
$$g(l,m,n) = (l-m-a-b+c)n - (l+m-a'-b)(l-m-a-b+c),$$

and we promise that $k_{l,m,n} = 0$ *if* l, m, *or* $n < 0$.

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