Noether's problem for some meta-abelian groups of small degree

By Akinari Hoshi

Department of Mathematical Sciences, Waseda University
3-4-1 Ohkubo, Shinjuku-ku, Tokyo 169-8555
(Communicated by Heisuke Hironaka, M. J. A., Jan. 12, 2005)

Abstract: In this note we solve Noether's problem over $\mathbf{Q}$ for some meta-abelian groups of small degree n. Let G be a subgroup of the group of one-dimensional affine transformations on $\mathbf{Z}/n\mathbf{Z}$ which contains $\mathbf{Z}/n\mathbf{Z}$. For $n = 9, 10, 12, 14, 15$, we show that Noether's problem for G has an affirmative answer by constructing an explicit transcendental basis of the fixed field over $\mathbf{Q}$.

Key words: Inverse Galois problem; generic polynomial; affine transformation group.

1. Introduction. Let $K = \mathbf{Q}(x_0, \ldots, x_{n-1})$ be the field of rational functions in n variables and G a transitive subgroup of S_n the symmetric group of degree n. Let G act on K by permuting the variables $x_0, \ldots, x_{n-1}$. Emmy Noether [11, 12] raised the following problem which is called Noether's problem for G (over $\mathbf{Q}$): Is the subfield K^G of G-invariant elements of K rational (i.e. purely transcendental) over $\mathbf{Q}$? This is one of central problems of the inverse Galois theory because if this problem has an affirmative answer then we get a $\mathbf{Q}$-generic polynomial for G (cf. [7]). The polynomial $g(\mathbf{t}; X) := g(t_1, \ldots, t_n; X) \in \mathbf{Q}(t_1, \ldots, t_n)[X]$, where $t_1, \ldots, t_n$ and X are indeterminates, is called $\mathbf{Q}$-generic for G if the splitting field of $g(\mathbf{t}; X)$ over $\mathbf{Q}(t_1, \ldots, t_n)$ has Galois group G and every Galois extension L/M with $\mathrm{Gal}(L/M) \cong G$ and $M \supset \mathbf{Q}$ is the splitting field of a polynomial $g(\mathbf{a}; X)$ for some $\mathbf{a} = (a_1, \ldots, a_n) \in M^n$. Namely every G-extension over a field M whose characteristic is zero can be obtained by some specialization of the parameters of $g(\mathbf{t}; X)$. In this note we shall solve Noether's problem for some meta-abelian groups G of small degree n by constructing an explicit transcendental basis of K^G over $\mathbf{Q}$. Let $\mathrm{Aff}(\mathbf{Z}/n\mathbf{Z})$ be the group of one-dimensional affine transformations on $\mathbf{Z}/n\mathbf{Z}$. We have $\mathrm{Aff}(\mathbf{Z}/n\mathbf{Z}) \cong (\mathbf{Z}/n\mathbf{Z}) \rtimes (\mathbf{Z}/n\mathbf{Z})^*$. The main result of this note is the following

Main Theorem. *Let G be a subgroup of $\mathrm{Aff}(\mathbf{Z}/n\mathbf{Z})$ containing $\mathbf{Z}/n\mathbf{Z}$. For $n = 9,10,12,14,15$, Noether's problem for G has an affirmative answer.*

We treated this problem in the cases $n \leq 7$ and $n = 11$ in [3, 4, 6]. In the previous paper [6], we extended Masuda's method [8, 9] for cyclic groups C_n, and we also use Masuda's approach in this note (cf. Lemma 1). Note that $n = 8$ is the smallest degree for which K^{C_n} is not rational over $\mathbf{Q}$. Moreover it is known that there does not exist $\mathbf{Q}$-generic polynomial for C_{8m}, and hence Noether's problem for C_{8m} has a negative answer (see [7]). Recently, however, it has been showed that K^{D_8}, K^{QD_8} and $K^{M_{16}}$ are rational over $\mathbf{Q}$, where D_8 (resp. QD_8, M_{16}) is the dihedral (resp. quasi-dihedral, modular) group of order 16 (see [1, 5]). The case $n = 13$ can not be applied original Masuda's approach as remarked by Endo-Miyata [2]. We shall treat some prime degree cases $n = p$ with $p \geq 13$ in a separate paper by studying the structure of the fixed field K^{C_p} in detail.

2. Preliminaries. In this section, we recall Masuda's method [8] for cyclic groups and its extension [6] for subgroups of $\mathrm{Aff}(\mathbf{Z}/n\mathbf{Z})$. Let σ be the cyclic permutation of the variables $x_0, \ldots, x_{n-1}$, i.e. $\sigma(x_0) = x_1, \ldots, \sigma(x_{n-1}) = x_0$ and τ_λ the x_0-fixed permutation defined by $\tau_\lambda(x_i) = x_{\lambda i}$ for $\lambda \in (\mathbf{Z}/n\mathbf{Z})^*$, where we take the subscript of x modulo n. We can identify a subgroup G of $\mathrm{Aff}(\mathbf{Z}/n\mathbf{Z})$ which contains $\mathbf{Z}/n\mathbf{Z}$ with $\langle \sigma, \tau_{\lambda_1}, \ldots, \tau_{\lambda_r} \rangle$ for certain $\lambda_1, \ldots, \lambda_r \in (\mathbf{Z}/n\mathbf{Z})^*$. For example, we have $D_n = \langle \sigma, \tau_{-1} \rangle$; the dihedral group of order $2n$. Let ζ be a primitive n-th root of unity, $\mathsf{k} := \mathbf{Q}(\zeta)$, $y_j := \sum_{i=0}^{n-1} \zeta^{-ij} x_i$, and $c_{j,k} := y_j y_k / y_{j+k}$ for $j, k = 0, \ldots, n-1$. We shall take the subscript of y and c modulo n, since $y_j = y_{mn+j}, (j = 0, \ldots, n-1)$. We have that $K^G(\zeta) = \mathsf{k}(x_0, \ldots, x_{n-1})^G$ for $G \subset S_n$ and $\mathsf{k}(x_0, \ldots, x_{n-1}) = \mathsf{k}(y_0, \ldots, y_{n-1})$. And we see that the actions of σ and τ_λ on the y_j's and the $c_{j,k}$'s are given by $\sigma(y_j) = \zeta^j y_j, \sigma(c_{j,k}) = c_{j,k}, \tau_\lambda(y_j) =$

2000 Mathematics Subject Classification. Primary 12F12; Secondary 11R32, 12F10.

$y_{\lambda^{-1}j}, \tau_\lambda(c_{j,k}) = c_{\lambda^{-1}j,\lambda^{-1}k}$, for $j,k = 0,\dots,n-1$. First we can obtain that

$$\mathsf{k}(x_0,\dots,x_{n-1})^{C_n} = \mathsf{k}(c_{j,k} \mid 0 \le j,k \le n-1),$$

and the $c_{j,k}$'s satisfy the following relations:

$$(1) \qquad c_{j,k} = \frac{c_{1,k}c_{1,k+1}\cdots c_{1,k+j-1}}{c_{1,1}c_{1,2}\cdots c_{1,j-1}}, \quad (j \ge 2).$$

Hence we have

$$(2) \qquad \mathsf{k}(x_0,\dots,x_{n-1})^{C_n} = \mathsf{k}(c_{1,0},c_{1,1},\dots,c_{1,n-1}).$$

Namely $\mathsf{k}(x_0,\dots,x_{n-1})^{C_n}$ is rational over k for any n. Masuda's method teaches us when we can descend the base field from k to $\mathbf{Q}$. For $f \in \mathsf{k}(x_0,\dots,x_{n-1})^{C_n}$, we define a set $[f]_{\mathrm{conj}} :=$ { all conjugates of f over K^{C_n}} and we put $\iota(f) := \#[f]_{\mathrm{conj}}$.

Lemma 1 (Masuda [8]). *Suppose that there exist elements* $a_1,\dots,a_t \in \mathsf{k}(x_0,\dots,x_{n-1})^{C_n}$ *such that* $\mathsf{k}(x_0,\dots,x_{n-1})^{C_n} = \mathsf{k}([a_i]_{\mathrm{conj}} \mid 1 \le i \le t)$ *and* $\sum_{i=1}^t \iota(a_i) = n$. *Let* $\omega_{i,1},\dots,\omega_{i,\iota(a_i)}$ *be a basis of* $\mathsf{k} \cap K^{C_n}(a_i)$ *over* $\mathbf{Q}$. *If we write* $a_i = \sum_{j=1}^{\iota(a_i)} \omega_{i,j} m_{j,i}$, $(m_{j,i} \in K^{C_n})$, *for* $i = 1,\dots,t$, *then* $K^{C_n} = \mathbf{Q}\big(m_{j,i} \mid 1 \le i \le t, 1 \le j \le \iota(a_i)\big)$.

Indeed, in the next section, we shall give such elements $a_1,\dots,a_t$ as in above lemma for $n = 9,10,12,14,15$ explicitly. For a subgroup $G = \langle \sigma, \tau_{\lambda_1},\dots,\tau_{\lambda_r}\rangle$ of $\mathrm{Aff}(\mathbf{Z}/n\mathbf{Z})$ containing $\mathbf{Z}/n\mathbf{Z}$, we have from Lemma 1 that

$$\begin{aligned} K^G &= (K^{C_n})^{G/C_n} = (K^{\langle\sigma\rangle})^{\langle \tau_{\lambda_1},\dots,\tau_{\lambda_r}\rangle} \\ &= \mathbf{Q}\big(m_{j,i} \bigm| 1 \le i \le t, 1 \le j \le \iota(a_i)\big)^{\langle \tau_{\lambda_1},\dots,\tau_{\lambda_r}\rangle}. \end{aligned}$$

We also can obtain the action of τ_λ on the transcendental basis $\{m_{j,i}\}$ of K^{C_n} over $\mathbf{Q}$ by using the equation $\tau_\lambda(c_{j,k}) = \alpha_{\lambda^{-1}}(c_{j,k})$ for $\lambda \in (\mathbf{Z}/n\mathbf{Z})^*$, where $\alpha_\lambda \in \mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$ such that $\alpha_\lambda(\zeta) = \zeta^\lambda$.

Let $x_0^{(j)},\dots,x_{n-1}^{(j)}, (j = 1,\dots,m)$ be variables and $L := K(x_0^{(1)},\dots,x_{n-1}^{(1)},\dots,x_0^{(m)},\dots,x_{n-1}^{(m)})$. It is well-known from the No-Name Lemma that if C_n acts on L as the cyclic permutation of the variables $x^{(j)} : x_0 \mapsto \cdots \mapsto x_{n-1} \mapsto x_0$, $x_0^{(j)} \mapsto \cdots \mapsto x_{n-1}^{(j)} \mapsto x_0^{(j)}$ for $j = 1,\dots,m$, then L^{C_n} is rational over K^{C_n} (cf. [10],[7, page 22]). Moreover we can give an explicit transcendental basis of L^{C_n} over K^{C_n}.

Lemma 2 ([6]). *We have*

$$\begin{aligned} L^{C_n} = K^{C_n}\big(&\mathrm{Tr}(x_0x_0^{(1)}),\dots,\mathrm{Tr}(x_0x_{n-1}^{(1)}), \\ &\dots,\mathrm{Tr}(x_0x_0^{(m)}),\dots,\mathrm{Tr}(x_0x_{n-1}^{(m)})\big), \end{aligned}$$

where Tr *is the trace under the action of* C_n.

Proof. The assertion follows from

$$\begin{aligned} L = K\big(&\mathrm{Tr}(x_0x_0^{(1)}),\dots,\mathrm{Tr}(x_0x_{n-1}^{(1)}), \\ &\dots,\mathrm{Tr}(x_0x_0^{(m)}),\dots,\mathrm{Tr}(x_0x_{n-1}^{(m)})\big), \end{aligned}$$

(see also [6]). □

3. Explicit transcendental basis of K^G.

We shall solve Noether's problem for subgroups G of $\mathrm{Aff}(\mathbf{Z}/n\mathbf{Z})$ containing $\mathbf{Z}/n\mathbf{Z}$ for each degree $n = 9,10,12,14,15$.

For $n = 9$, the subgroups of $\mathrm{Aff}(\mathbf{Z}/9\mathbf{Z})$ containing $\mathbf{Z}/9\mathbf{Z}$ are $C_9 = \langle\sigma\rangle$, $D_9 = \langle \sigma, \tau_{-1}\rangle$ ($\cong C_9 \rtimes C_2$), $G_{9,3} := \langle \sigma, \tau_4\rangle$ ($\cong C_9 \rtimes C_3$), $\mathrm{Aff}(\mathbf{Z}/9\mathbf{Z}) = \langle \sigma, \tau_2\rangle$ ($\cong C_9 \rtimes C_6$).

Proposition 1. *We have*

$$\begin{aligned} &\mathsf{k}(x_0,\dots,x_8)^{C_9} \\ &= \mathsf{k}(c_{0,1}, [c_{1,2}+c_{4,8}+c_{5,7}]_{\mathrm{conj}}, [c_{1,4}]_{\mathrm{conj}}). \end{aligned}$$

Proof. We see easily that $c_{0,1} = y_0 = x_0 + \cdots + x_8 \in K^{C_9}$, $[c_{1,2}]_{\mathrm{conj}} = \{c_{1,2},c_{2,4},c_{4,8},c_{1,5},c_{5,7},c_{7,8}\}$ and $[c_{1,4}]_{\mathrm{conj}} = \{c_{1,4},c_{2,8},c_{4,7},c_{2,5},c_{1,7},c_{5,8}\}$. By using (1), we can obtain that

$$c_{1,1} = \frac{c_{1,6}c_{7,8}}{c_{2,8}}, \quad c_{1,3} = \frac{c_{1,6}c_{7,8}}{c_{4,8}}, \quad c_{1,5} = \frac{c_{2,5}c_{7,8}}{c_{2,8}},$$

$$c_{1,6} = \frac{c_{1,2}c_{2,5}c_{4,7}}{c_{2,4}c_{5,7}}, \quad c_{1,7} = \frac{c_{1,2}c_{4,7}}{c_{4,8}}, \quad c_{1,8} = \frac{c_{1,6}c_{7,8}}{c_{0,1}}.$$

Therefore it follows from (2) that $\mathsf{k}(x_0,\dots,x_8)^{C_9} = \mathsf{k}(c_{0,1},[c_{1,2}]_{\mathrm{conj}},[c_{1,4}]_{\mathrm{conj}})$. And we have the following relations:

$$c_{1,5}c_{2,8} - c_{2,5}c_{7,8} = 0, \quad c_{2,4}c_{5,8} - c_{2,5}c_{7,8} = 0,$$
$$c_{1,2}c_{4,7} - c_{1,7}c_{4,8} = 0, \quad c_{1,2}c_{4,7} - c_{1,4}c_{5,7} = 0.$$

Put $u_{1,2} := c_{1,2}+c_{4,8}+c_{5,7}$, $u_{1,5} := c_{1,5}+c_{2,4}+c_{7,8}$, then we have from above equations that

$$c_{4,8} = \frac{c_{1,4}c_{4,7}u_{1,2}}{W_1}, \qquad c_{5,7} = \frac{c_{1,7}c_{4,7}u_{1,2}}{W_1},$$
$$c_{2,4} = \frac{c_{2,5}c_{2,8}u_{1,5}}{W_2}, \qquad c_{7,8} = \frac{c_{2,8}c_{5,8}u_{1,5}}{W_2},$$

where $W_1 = c_{1,4}c_{1,7} + c_{1,4}c_{4,7} + c_{1,7}c_{4,7}$, $W_2 = c_{2,5}c_{2,8}+c_{2,5}c_{5,8}+c_{2,8}c_{5,8}$. The assertion follows from $\mathsf{k}(x_0,\dots,x_8)^{C_9} = \mathsf{k}(c_{0,1},u_{1,2},u_{1,5},[c_{1,4}]_{\mathrm{conj}})$. □

Thus it follows from Lemma 1 that

$$K^{C_9} = \mathbf{Q}(y_0, s_1', s_2', t_1', t_2', \dots, t_6'),$$

where

$$\begin{aligned} c_{1,2}+c_{4,8}+c_{5,7} &= s_1'\zeta^3 + s_2'\zeta^6, \\ c_{1,4} &= t_1'\zeta + t_2'\zeta^2 + \cdots + t_6'\zeta^6. \end{aligned}$$

We see that the τ_2-action on K^{C_9} is given by

$$y_0 \mapsto y_0,\ s'_1 \leftrightarrow s'_2,$$
$$t'_1 \mapsto t'_2 - t'_5,\ t'_2 \mapsto t'_4,\ t'_3 \leftrightarrow t'_6,\ t'_4 \mapsto -t'_5,\ t'_5 \mapsto t'_1,$$

since $\tau_2(c_{1,4}) = \alpha_5(c_{1,4})$, where $\alpha_5(\zeta) = \zeta^5$. We use the following non-singular transformation

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{bmatrix} := \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} t'_1 \\ t'_2 \\ t'_3 \\ t'_4 \\ t'_5 \\ t'_6 \end{bmatrix}.$$

Then we have

$$K^{C_9} = \mathbf{Q}(y_0, s'_1, s'_2, t_1, t_2, \ldots, t_6),$$

and the τ_2-action on it can be described as

$$y_0 \mapsto y_0,\ s'_1 \leftrightarrow s'_2,\ t_1 \mapsto t_2 \mapsto \cdots \mapsto t_6 \mapsto t_1.$$

We see that $s'_1 = f_1^{(3)}/f_3^{(2)}$, $s'_2 = f_2^{(3)}/f_3^{(2)}$, $t_i = g_i^{(7)}/g_7^{(6)}$, where $f_j^{(k)}$, $g_i^{(k)}$ are homogeneous elements of degree k in $\mathbf{Q}[x_0, \ldots, x_8]$ for $j = 1, 2, 3$ and $i = 1, \ldots, 7$. From Lemma 2, we put

$$s_1 := s'_1(t_1 + t_3 + t_5) + s'_2(t_2 + t_4 + t_6),$$
$$s_2 := s'_1(t_2 + t_4 + t_6) + s'_2(t_1 + t_3 + t_5),$$

then we see that s_1, s_2 are τ_2-invariants, i.e. $\mathrm{Aff}(\mathbf{Z}/9\mathbf{Z})$-invariants, and we have

$$K^{C_9} = \mathbf{Q}(y_0, s_1, s_2, t_1, t_2, \ldots, t_6).$$

For D_9 and $G_{9,3} = \langle \sigma, \tau_4 \rangle$, since the τ_{-1}-action (resp. τ_4-action) on K^{C_9} above is given by $t_1 \leftrightarrow t_4$, $t_2 \leftrightarrow t_5$, $t_3 \leftrightarrow t_6$ (resp. $t_1 \mapsto t_3 \mapsto t_5 \mapsto t_1$, $t_2 \mapsto t_4 \mapsto t_6 \mapsto t_2$), we have from Lemma 2 that

$$\begin{aligned} K^{D_9} &= \left(K^{C_9}\right)^{\langle \tau_{-1} \rangle} \\ &= \mathbf{Q}\big(y_0, s_1, s_2, t_1 + t_4, t_1t_4, t_1t_2 + t_4t_5, \\ &\qquad t_1t_5 + t_2t_4, t_1t_3 + t_4t_6, t_1t_6 + t_3t_4\big), \\ K^{G_{9,3}} &= \left(K^{C_9}\right)^{\langle \tau_4 \rangle} \\ &= \mathbf{Q}\left(y_0, s_1, s_2, t_1 + t_3 + t_5, \frac{\mathrm{Nr}(t_1 + t_3 - 2t_5)}{\mathrm{Tr}(t_1^2 - t_1t_3)},\right. \\ &\qquad \left.\frac{\mathrm{Nr}(t_1 - t_3)}{\mathrm{Tr}(t_1^2 - t_1t_3)}, \mathrm{Tr}(t_1t_2), \mathrm{Tr}(t_1t_4), \mathrm{Tr}(t_1t_6)\right), \end{aligned}$$

where Nr and Tr are the norm and the trace under the action of τ_4. Because it is well-known a transcendental basis of $\mathbf{Q}(x_0, \ldots, x_5)^{C_6} = \mathbf{Q}(z_1, \ldots, z_6)$ over $\mathbf{Q}$ (see, for example, [6]), we can obtain an explicit transcendental basis of $K^{\mathrm{Aff}(\mathbf{Z}/9\mathbf{Z})}$ over $\mathbf{Q}$ by using $z_1, \ldots, z_6$.

For $n = 10$, we see that the subgroups of $\mathrm{Aff}(\mathbf{Z}/10\mathbf{Z})$ containing $\mathbf{Z}/10\mathbf{Z}$ are $C_{10} = \langle \sigma \rangle$, $D_{10} = \langle \sigma, \tau_{-1} \rangle$, $\mathrm{Aff}(\mathbf{Z}/10\mathbf{Z}) = \langle \sigma, \tau_3 \rangle$ ($\cong C_{10} \rtimes C_4$). In the previous paper [6], we showed the following

Proposition 2 ([6]). *We have*

$$\begin{aligned} &\mathsf{k}(x_0, \ldots, x_9)^{C_{10}} \\ &= \mathsf{k}(c_{0,1}, [c_{1,4}]_{\mathrm{conj}}, [c_{1,8}]_{\mathrm{conj}}, c_{1,9} + c_{3,7}). \end{aligned}$$

Hence, by applying Lemma 1 to Proposition 2, we have

$$K^{C_{10}} = \mathbf{Q}(y_0, r_1, s'_1, \ldots, s'_4, t_1, \ldots, t_4),$$

where

$$\begin{aligned} r_1 &= c_{1,9} + c_{3,7}, \\ c_{1,4} &= s'_1\zeta + s'_2\zeta^3 + s'_4\zeta^7 + s'_3\zeta^9, \\ c_{1,8} &= t_1\zeta + t_2\zeta^3 + t_4\zeta^7 + t_3\zeta^9. \end{aligned}$$

And the action of τ_3 on it is given by

$$\begin{aligned} &y_0 \mapsto y_0,\ r_1 \mapsto r_1, \\ &s'_1 \mapsto s'_2 \mapsto s'_3 \mapsto s'_4 \mapsto s'_1, \\ &t_1 \mapsto t_2 \mapsto t_3 \mapsto t_4 \mapsto t_1. \end{aligned}$$

We also see that $s'_i = f_i^{(5)}/f_5^{(4)}$, $t_j = g_j^{(5)}/g_5^{(4)}$, where $f_i^{(k)}$, $g_j^{(k)}$ are homogeneous elements of degree k in $\mathbf{Q}[x_0, \ldots, x_9]$ for $i, j = 1, \ldots, 5$. From Lemma 2, we put $s_i := \mathrm{Tr}(s'_1t_i)$ for $i = 1, \ldots, 4$, where Tr is the trace under the action of τ_3, then we see that $s_1, \ldots, s_4$ are τ_3-invariants (i.e. $\mathrm{Aff}(\mathbf{Z}/10\mathbf{Z})$-invariants) and we have

$$K^{C_{10}} = \mathbf{Q}(y_0, r_1, s_1, \ldots, s_4, t_1, \ldots, t_4).$$

Therefore if we put $u_1 := t_1 + t_3$, $u_2 := t_2 + t_4$, $v_1 := t_1 - t_3$, $v_2 := t_2 - t_4$, then we get

$$\begin{aligned} K^{D_{10}} &= \mathbf{Q}\big(y_0, r_1, s_1, s_2, s_3, s_4, t_1 + t_3, \\ &\qquad t_1t_3, t_1t_2 + t_3t_4, t_1t_4 + t_2t_3\big), \\ K^{\mathrm{Aff}(\mathbf{Z}/10\mathbf{Z})} &= \mathbf{Q}\big(y_0, r_1, s_1, \ldots, s_4, u_1 + u_2, v_1^2 + v_2^2, \\ &\qquad (u_1 - u_2)v_1v_2, (u_1 - u_2)(v_1^2 - v_2^2)\big). \end{aligned}$$

For $n = 12$, the groups $C_{12} = \langle \sigma \rangle$, $D_{12} = \langle \sigma, \tau_{-1} \rangle$, $G_{12,2}^{(1)} = \langle \sigma, \tau_5 \rangle$, $G_{12,2}^{(2)} = \langle \sigma, \tau_7 \rangle$ and $\mathrm{Aff}(\mathbf{Z}/12\mathbf{Z}) = \langle \sigma, \tau_{-1}, \tau_5 \rangle$ ($\cong C_{12} \rtimes (C_2 \times C_2)$) are subgroups of $\mathrm{Aff}(\mathbf{Z}/12\mathbf{Z})$ which contain $\mathbf{Z}/12\mathbf{Z}$.

Proposition 3. *We have*

$$\begin{aligned} \mathsf{k}(x_0, \ldots, x_{11})^{C_{12}} &= \mathsf{k}(c_{0,1}, [c_{1,2} + c_{5,10}]_{\mathrm{conj}}, [c_{1,3}]_{\mathrm{conj}}, \\ &\qquad [c_{1,5}]_{\mathrm{conj}}, [c_{1,7}]_{\mathrm{conj}}, c_{1,11} + c_{5,7}). \end{aligned}$$

Proof. We have that $c_{0,1} = y_0 = x_0 + \cdots + x_{11} \in K^{C_{12}}$, $[c_{1,2}]_{\mathrm{conj}} = \{c_{1,2}, c_{5,10}, c_{2,7}, c_{10,11}\}$, $[c_{1,3}]_{\mathrm{conj}} = \{c_{1,3}, c_{3,5}, c_{7,9}, c_{9,11}\}$, $[c_{1,5}]_{\mathrm{conj}} = \{c_{1,5}, c_{7,11}\}$, $[c_{1,7}]_{\mathrm{conj}} = \{c_{1,7}, c_{5,11}\}$ and $[c_{1,11}]_{\mathrm{conj}} = \{c_{1,11}, c_{5,7}\}$. By using (1), we can obtain that

$$c_{1,1} = \frac{c_{0,1}c_{1,7}c_{1,11}}{c_{2,7}c_{9,11}}, \ c_{1,4} = \frac{c_{0,1}c_{1,11}}{c_{5,11}}, \ c_{1,6} = \frac{c_{0,1}c_{1,11}}{c_{7,11}},$$
$$c_{1,8} = \frac{c_{0,1}c_{1,11}}{c_{9,11}}, \ c_{1,9} = \frac{c_{0,1}c_{1,11}}{c_{10,11}}, \ c_{1,10} = \frac{c_{1,3}c_{5,10}}{c_{5,11}}.$$

Hence it follows from (2) that $\mathsf{k}(x_0, \ldots, x_{11})^{C_{12}} = \mathsf{k}(c_{0,1}, [c_{1,i}]_{\mathrm{conj}} \mid i = 2, 3, 5, 7, 11)$ and we also have

$$c_{1,3}c_{5,10}c_{7,11} - c_{1,5}c_{7,9}c_{10,11} = 0,$$
$$c_{1,2}c_{5,7}c_{10,11} - c_{1,11}c_{2,7}c_{5,10} = 0,$$
$$c_{1,3}c_{5,7}c_{9,11} - c_{1,11}c_{3,5}c_{7,9} = 0.$$

Put $u_{1,2} := c_{1,2} + c_{5,10}$, $u_{2,7} := c_{2,7} + c_{10,11}$, $u_{1,11} := c_{1,11} + c_{5,7}$, then we have from above equations that

$$c_{5,10} = \frac{c_{7,9}(c_{3,5}c_{7,11}u_{1,2} - c_{1,5}c_{9,11}u_{2,7})}{c_{7,11}(c_{3,5}c_{7,9} - c_{1,3}c_{9,11})},$$
$$c_{10,11} = \frac{c_{1,3}(c_{3,5}c_{7,11}u_{1,2} - c_{1,5}c_{9,11}u_{2,7})}{c_{1,5}(c_{3,5}c_{7,9} - c_{1,3}c_{9,11})},$$
$$c_{5,7} = \frac{c_{3,5}c_{7,9}u_{1,11}}{c_{3,5}c_{7,9} + c_{1,3}c_{9,11}}.$$

Thus the assertion follows from $\mathsf{k}(x_0, \ldots, x_{11})^{C_{12}} = \mathsf{k}(c_{0,1}, u_{1,2}, u_{2,7}, [c_{1,3}]_{\mathrm{conj}}, [c_{1,5}]_{\mathrm{conj}}, [c_{1,7}]_{\mathrm{conj}}, u_{1,11})$. □

By applying Lemma 1 to Proposition 3, we have

$$K^{C_{12}} = \mathbf{Q}(y_0, r_1, s_1', s_2', s_3', s_4', t_1, t_2, u_1', u_2', v_1, v_2),$$

where

$$r_1 = c_{1,11} + c_{5,7},$$
$$c_{1,3} = s_1'\zeta + s_2'\zeta^2 + s_3'\zeta^3 + s_4'\zeta^4,$$
$$c_{1,5} = t_1(\zeta^2 - \zeta^4) + t_2\zeta^3,$$
$$c_{1,7} = u_1'\zeta^2 + u_2'\zeta^4,$$
$$c_{1,2} + c_{5,10} = v_1(\zeta^2 - \zeta^4) + v_2\zeta^3.$$

Since $\zeta^2 - \zeta^4 = 1$, we have that y_0, r_1, t_1, v_1 are $\mathrm{Aff}(\mathbf{Z}/12\mathbf{Z})$-invariants. From the equation $\tau_\lambda(c_{j,k}) = \alpha_{\lambda^{-1}}(c_{j,k})$, we obtain that the action of τ_{-1} (resp. τ_5, τ_7) on $K^{C_{12}}$ above is given as follows: $(s_1', s_2', s_3', s_4', t_2, u_1', u_2', v_2) \mapsto (s_1', -s_4', -(s_1' + s_3'), -s_2', -t_2, -u_2', -u_1', -v_2)$, (resp. $(-s_1', -s_4', s_1' + s_3', -s_2', t_2, -u_2', -u_1', v_2)$, $(-s_1, s_2, -s_3, s_4, -t_2, u_1', u_2', -v_2)$). Now we use the following bi-rational transformation:

$$\begin{cases} s_1 := s_1', \\ s_2 := s_2' + s_4', \\ s_3 := s_1' + 2s_3', \\ s_4 := s_2' - s_4', \\ u_1 := u_1' + u_2', \\ u_2 := u_1' - u_2', \end{cases} \qquad \begin{cases} s_1' = s_1, \\ s_2' = (s_2 + s_4)/2, \\ s_3' = (-s_1 + s_3)/2, \\ s_4' = (s_2 - s_4)/2, \\ u_1' = (u_1 + u_2)/2, \\ u_2' = (u_1 - u_2)/2. \end{cases}$$

Then we have that s_4, u_2 are $\mathrm{Aff}(\mathbf{Z}/12\mathbf{Z})$-invariants and $\tau_{-1} : (s_1, s_2, s_3, u_1) \mapsto (s_1, -s_2, -s_3, -u_1)$, $\tau_5 : (s_1, s_2, s_3, u_1) \mapsto (-s_1, -s_2, s_3, -u_1)$, $\tau_7 = \tau_{-1}\tau_5$. We put $W := (y_0, r_1, s_4, t_1, u_2, v_1)$ then $K^{C_{12}} = \mathbf{Q}(W, s_1, s_2, s_3, t_2, u_1, v_2)$, and hence we get

$$K^{D_{12}} = \mathbf{Q}\Big(W, s_1, s_2^2, \frac{s_3}{s_2}, \frac{t_2}{s_2}, \frac{u_1}{s_2}, \frac{v_2}{s_2}\Big),$$
$$K^{G_{12,2}^{(1)}} = \mathbf{Q}\Big(W, s_1^2, \frac{s_2}{s_1}, s_3, t_2, \frac{u_1}{s_1}, v_2\Big),$$
$$K^{G_{12,2}^{(2)}} = \mathbf{Q}\Big(W, s_1^2, s_2, \frac{s_3}{s_1}, \frac{t_2}{s_1}, u_1, \frac{v_2}{s_1}\Big).$$

Since τ_5 acts on $K^{D_{12}}$ as

$$\Big(s_1, s_2^2, \frac{s_3}{s_2}, \frac{t_2}{s_2}, \frac{u_1}{s_2}, \frac{v_2}{s_2}\Big) \mapsto \Big(-s_1, s_2^2, -\frac{s_3}{s_2}, -\frac{t_2}{s_2}, \frac{u_1}{s_2}, -\frac{v_2}{s_2}\Big),$$

we have

$$K^{\mathrm{Aff}(\mathbf{Z}/12\mathbf{Z})} = \mathbf{Q}\Big(W, s_1^2, s_2^2, \frac{s_3}{s_1s_2}, \frac{t_2}{s_1s_2}, \frac{u_1}{s_2}, \frac{v_2}{s_1s_2}\Big).$$

For $n = 14$, we have that the subgroups of $\mathrm{Aff}(\mathbf{Z}/14\mathbf{Z})$ containing $\mathbf{Z}/14\mathbf{Z}$ are $C_{14} = \langle\sigma\rangle$, $D_{14} = \langle\sigma, \tau_{-1}\rangle$, $G_{14,3} := \langle\sigma, \tau_9\rangle$ $(\cong C_{14} \rtimes C_3)$, $\mathrm{Aff}(\mathbf{Z}/14\mathbf{Z}) = \langle\sigma, \tau_3\rangle$ $(\cong C_{14} \rtimes C_6)$.

Proposition 4. *We have*

$$\mathsf{k}(x_0, \ldots, x_{13})^{C_{14}} = \mathsf{k}(c_{0,1}, [c_{1,6}]_{\mathrm{conj}}, [c_{1,12}]_{\mathrm{conj}}, c_{1,13} + c_{3,11} + c_{5,9}).$$

Proof. We have that $c_{0,1} = y_0 = x_0 + \cdots + x_{13} \in K^{C_{14}}$, $[c_{1,6}]_{\mathrm{conj}} = \{c_{1,6}, c_{3,4}, c_{2,5}, c_{9,12}, c_{10,11}, c_{8,13}\}$, $[c_{1,12}]_{\mathrm{conj}} = \{c_{1,12}, c_{3,8}, c_{4,5}, c_{9,10}, c_{6,11}, c_{2,13}\}$ and $[c_{1,13}]_{\mathrm{conj}} = \{c_{1,13}, c_{3,11}, c_{5,9}\}$. By using (1), we can obtain that

$$c_{1,1} = \frac{c_{0,1}c_{1,13}}{c_{2,13}}, \ c_{1,2} = \frac{c_{1,5}c_{2,5}c_{6,11}c_{9,10}}{c_{0,1}c_{5,9}c_{10,11}},$$
$$c_{1,3} = \frac{c_{0,1}c_{1,13}c_{3,8}c_{10,11}}{c_{4,5}c_{8,13}c_{9,10}}, \ c_{1,4} = \frac{c_{1,5}c_{3,4}c_{6,11}c_{9,10}}{c_{0,1}c_{5,9}c_{10,11}},$$
$$c_{1,5} = \frac{c_{0,1}c_{1,13}c_{2,5}}{c_{1,6}c_{2,13}}, \ c_{1,7} = \frac{c_{0,1}c_{1,13}}{c_{8,13}},$$
$$c_{1,8} = \frac{c_{1,5}c_{3,8}c_{6,11}}{c_{0,1}c_{5,9}}, \ c_{1,9} = \frac{c_{1,5}c_{3,4}c_{6,11}}{c_{4,5}c_{10,11}},$$

$$c_{1,10} = \frac{c_{1,13}c_{3,8}c_{10,11}}{c_{3,11}c_{8,13}}, \quad c_{1,11} = \frac{c_{1,5}c_{3,4}c_{6,11}}{c_{4,5}c_{9,12}}.$$

Thus it follows from (2) that $\mathsf{k}(x_0,\ldots,x_{13})^{C_{14}} = \mathsf{k}(c_{0,1},[c_{1,6}]_{\mathrm{conj}},[c_{1,12}]_{\mathrm{conj}},[c_{1,13}]_{\mathrm{conj}})$. We also get the following two relations:

$$c_{1,6}c_{3,11}c_{4,5}c_{8,13}c_{9,10} - c_{1,13}c_{3,4}c_{3,8}c_{6,11}c_{10,11} = 0,$$
$$c_{2,5}c_{3,11}c_{4,5}c_{9,10}c_{9,12} - c_{1,12}c_{2,13}c_{3,4}c_{5,9}c_{10,11} = 0.$$

Put $r_1 := c_{1,13} + c_{3,11} + c_{5,9}$, then we have

$$c_{3,11} = \frac{c_{1,12}c_{2,13}c_{3,4}c_{3,8}c_{6,11}c_{10,11}r_1}{c_{1,6}c_{8,13}v_1v_2 + c_{2,5}c_{9,12}v_2v_3 + c_{3,4}c_{10,11}v_1v_3},$$
$$c_{5,9} = \frac{c_{2,5}c_{3,8}c_{4,5}c_{6,11}c_{9,10}c_{9,12}r_1}{c_{1,6}c_{8,13}v_1v_2 + c_{2,5}c_{9,12}v_2v_3 + c_{3,4}c_{10,11}v_1v_3},$$

where $v_1 = c_{1,12}c_{2,13}, v_2 = c_{4,5}c_{9,10}, v_3 = c_{3,8}c_{6,11}$. Hence the assertion follows. □

From Lemma 1 we have

$$K^{C_{14}} = \mathbf{Q}(y_0, r_1, s'_1, \ldots, s'_6, t_1, \ldots, t_6),$$

where

$$r_1 = c_{1,13} + c_{3,11} + c_{5,9},$$
$$c_{1,6} = s'_1\zeta + s'_2\zeta^3 + s'_3\zeta^9 + s'_4\zeta^{13} + s'_5\zeta^{11} + s'_6\zeta^5,$$
$$c_{1,12} = t_1\zeta + t_2\zeta^3 + t_3\zeta^9 + t_4\zeta^{13} + t_5\zeta^{11} + t_6\zeta^5,$$

and the τ_3-action on it is given by

$$y_0 \mapsto y_0, \ r_1 \mapsto r_1,$$
$$s'_1 \mapsto s'_2 \mapsto \cdots \mapsto s'_6 \mapsto s'_1,$$
$$t_1 \mapsto t_2 \mapsto \cdots \mapsto t_6 \mapsto t_1,$$

since $\tau_3(c_{1,k}) = \alpha_5(c_{1,k})$ for $k = 6, 12$. From Lemma 2, we put $s_i := \mathrm{Tr}(s'_1 t_i)$ for $i = 1, \ldots, 6$, where Tr is the trace under the action of τ_3, then we see that $s_1, \ldots, s_6$ are $\mathrm{Aff}(\mathbf{Z}/14\mathbf{Z})$-invariants and

$$K^{C_{14}} = \mathbf{Q}(y_0, r_1, s_1, \ldots, s_6, t_1, \ldots, t_6).$$

Therefore we obtain an explicit transcendental basis of $K^{D_{14}}, K^{G_{14,3}}$ and $K^{\mathrm{Aff}(\mathbf{Z}/14\mathbf{Z})}$ by using the same manner as in the case $n = 9$.

For $n = 15$, we see that the subgroups of $\mathrm{Aff}(\mathbf{Z}/15\mathbf{Z})$ containing $\mathbf{Z}/15\mathbf{Z}$ are $C_{15} = \langle\sigma\rangle$, $D_{15} = \langle\sigma, \tau_{-1}\rangle$, $G^{(1)}_{15,2} := \langle\sigma, \tau_4\rangle$, $(\cong C_{15} \rtimes C_2 \cong C_5 \rtimes C_6 \cong D_5 \times C_3)$, $G^{(2)}_{15,2} := \langle\sigma, \tau_{11}\rangle$, $(\cong C_{15} \rtimes C_2 \cong S_3 \times C_5)$, $G_{15,2,2} := \langle\sigma, \tau_{-1}, \tau_4\rangle$ $(\cong C_{15} \rtimes (C_2 \times C_2))$, $G_{15,4} := \langle\sigma, \tau_2\rangle$ $(\cong C_{15} \rtimes C_4)$, $\mathrm{Aff}(\mathbf{Z}/15\mathbf{Z}) = \langle\sigma, \tau_{-1}, \tau_2\rangle$ $(\cong C_{15} \rtimes (C_2 \times C_4))$. We note that there are precisely 4 groups of order thirty (i.e. $C_{30}, D_{15}, G^{(1)}_{15,2}, G^{(2)}_{15,2}$), (cf. [13]).

Proposition 5. *We have*

$$\mathsf{k}(x_0, \ldots, x_{14})^{C_{15}} = \mathsf{k}(c_{0,1}, [c_{1,4} + c_{11,14}]_{\mathrm{conj}}, [c_{1,7}]_{\mathrm{conj}}, [c_{1,11}]_{\mathrm{conj}}).$$

Proof. We have that $c_{0,1} = y_0 = x_0 + \cdots + x_{14} \in K^{C_{15}}$, $[c_{1,4}]_{\mathrm{conj}} = \{c_{1,4}, c_{2,8}, c_{7,13}, c_{11,14}\}$, $[c_{1,7}]_{\mathrm{conj}} = \{c_{1,7}, c_{2,14}, c_{4,13}, c_{4,7}, c_{8,11}, c_{2,11}, c_{1,13}, c_{8,14}\}$ and $[c_{1,11}]_{\mathrm{conj}} = \{c_{1,11}, c_{2,7}, c_{4,14}, c_{8,13}\}$. By using (1), we can obtain that

$$c_{1,1} = \frac{c_{1,7}c_{8,14}}{c_{2,14}}, \quad c_{1,2} = \frac{c_{1,13}c_{4,14}}{c_{4,13}}, \quad c_{1,3} = \frac{c_{1,7}c_{8,14}}{c_{4,14}},$$
$$c_{1,5} = \frac{c_{1,7}c_{8,13}}{c_{7,13}}, \quad c_{1,6} = \frac{c_{1,13}c_{8,14}}{c_{8,13}}, \quad c_{1,8} = \frac{c_{2,7}c_{8,14}}{c_{2,14}},$$
$$c_{1,9} = \frac{c_{1,7}c_{2,8}}{c_{2,7}}, \quad c_{1,10} = \frac{c_{1,7}c_{8,14}}{c_{11,14}},$$
$$c_{1,12} = \frac{c_{1,7}c_{2,11}c_{8,14}}{c_{1,11}c_{2,14}}, \quad c_{1,14} = \frac{c_{1,7}c_{8,14}}{c_{0,1}}.$$

Therefore from (2) we have $\mathsf{k}(x_0, \ldots, x_{14})^{C_{15}} = \mathsf{k}(c_{0,1}, [c_{1,4}]_{\mathrm{conj}}, [c_{1,7}]_{\mathrm{conj}}, [c_{1,11}]_{\mathrm{conj}})$. And we can obtain the following two relations:

$$c_{1,7}c_{2,8}c_{4,13} - c_{1,13}c_{4,7}c_{11,14} = 0,$$
$$c_{1,4}c_{2,14}c_{8,11} - c_{2,11}c_{7,13}c_{8,14} = 0.$$

Put $u_{1,4} := c_{1,4} + c_{11,14}, u_{2,8} = c_{2,8} + c_{7,13}$, then we have

$$c_{11,14} = \frac{c_{1,7}c_{4,13}(c_{2,14}c_{8,11}u_{1,4} - c_{2,11}c_{8,14}u_{2,8})}{c_{1,7}c_{2,14}c_{4,13}c_{8,11} - c_{1,13}c_{2,11}c_{4,7}c_{8,14}},$$
$$c_{7,13} = -\frac{c_{2,14}c_{8,11}(c_{1,13}c_{4,7}u_{1,4} - c_{1,7}c_{4,13}u_{2,8})}{c_{1,7}c_{2,14}c_{4,13}c_{8,11} - c_{1,13}c_{2,11}c_{4,7}c_{8,14}}.$$

This proves the assertion. □

From Lemma 1 we have

$$K^{C_{15}} = \mathbf{Q}(y_0, r'_1, r'_2, s'_1, \ldots, s'_4, t'_1, \ldots, t'_4, u_1, \ldots, u_4),$$

where

$$c_{1,4} + c_{11,14} = r'_1(\zeta + \zeta^4 + \zeta^{11} + \zeta^{14}) + r'_2(\zeta^2 + \zeta^7 + \zeta^8 + \zeta^{13}),$$
$$c_{1,7} = s'_1\zeta + s'_2\zeta^2 + s'_3\zeta^4 + s'_4\zeta^8 + t'_1\zeta^{14} + t'_2\zeta^{13} + t'_3\zeta^{11} + t'_4\zeta^7,$$
$$c_{1,11} = u_1(\zeta + \zeta^{11}) + u_2(\zeta^2 + \zeta^7) + u_3(\zeta^4 + \zeta^{14}) + u_4(\zeta^8 + \zeta^{13}).$$

And the actions of τ_{-1} and τ_2 on $K^{C_{15}}$ are given by

$$\tau_{-1} : y_0 \mapsto y_0, \ r'_1 \mapsto r'_1, \ r'_2 \mapsto r'_2 \ s'_1 \leftrightarrow t'_1,$$
$$s'_2 \leftrightarrow t'_2, \ s'_3 \leftrightarrow t'_3, \ s'_4 \leftrightarrow t'_4, \ u_1 \leftrightarrow u_3, \ u_2 \leftrightarrow u_4,$$
$$\tau_2 : y_0 \mapsto y_0, \ r'_1 \leftrightarrow r'_2 \ s'_1 \mapsto s'_2 \mapsto s'_3 \mapsto s'_4 \mapsto s'_1,$$
$$t'_1 \mapsto t'_2 \mapsto t'_3 \mapsto t'_4 \mapsto t'_1, u_1 \mapsto u_2 \mapsto u_3 \mapsto u_4 \mapsto u_1.$$

We also have $\tau_4 = \tau_2^2, \tau_{11} = \tau_{-1}\tau_4$. Using Lemma 2, we put $r_1 := \mathrm{Tr}(r_1' u_1)$, $r_2 := \mathrm{Tr}(r_2' u_1)$, $s_i := \mathrm{Tr}(s_i' u_1)$, $t_i := \mathrm{Tr}(t_i' u_1)$ for $i = 1, \dots, 4$, where Tr is the trace under the action of τ_2. We see that y_0, r_1, r_2 are $\mathrm{Aff}(\mathbf{Z}/15\mathbf{Z})$-invariants. Hence we put $W = (y_0, r_1, r_2)$ then we have

$$K^{C_{15}} = \mathbf{Q}(W, s_1, \dots, s_4, t_1, \dots, t_4, u_1, \dots, u_4).$$

The τ_{-1}-action on $K^{C_{15}}$ above is given by $s_1 \leftrightarrow t_1$, $s_2 \leftrightarrow t_2$, $s_3 \leftrightarrow t_3$, $s_4 \leftrightarrow t_4$, $u_1 \leftrightarrow u_3$, $u_2 \leftrightarrow u_4$ and τ_2 acts on $s_1, \dots, s_4, t_1, \dots, t_4$ trivially and the u_i's as $u_1 \mapsto u_2 \mapsto u_3 \mapsto u_4 \mapsto u_1$. Therefore we can easily obtain an explicit transcendental basis of $K^{D_{15}}$, $K^{G_{15,2}^{(1)}}$, $K^{G_{15,2}^{(2)}}$, $K^{G_{15,2,2}}$, $K^{G_{15,4}}$ and $K^{\mathrm{Aff}(\mathbf{Z}/15\mathbf{Z})}$ using the same way as in the case $n = 10$.

Acknowledgements. The author is a Research Fellow of the Japan Society for the Promotion of Science and supported by Grant-in-Aid for Scientific Research for JSPS Fellows. He also would like to express his gratitude to Professor Ki-ichiro Hashimoto who gave him various suggestions during this study.

References

[1] H. Chu, S.-J. Hu and M. Kang, Noether's problem for dihedral 2-groups, Comment. Math. Helv. **79** (2004), no. 1, 147–159.

[2] S. Endô and T. Miyata, Invariants of finite abelian groups, J. Math. Soc. Japan **25** (1973), 7–26.

[3] K. Hashimoto and A. Hoshi, Families of cyclic polynomials obtained from geometric generalization of Gaussian period relations, Math. Comp. (To appear).

[4] K. Hashimoto and A. Hoshi, Geometric generalization of Gaussian period relations with application to Noether's problem for meta-cyclic groups, Tokyo J. Math. (To appear).

[5] K. Hashimoto, A. Hoshi and Y. Rikuna, Noether's problem and **Q**-generic polynomials for the normalizer of the 8-cycle in S_8, (2004). (Preprint).

[6] A. Hoshi, Noether's problem for Frobenius groups of degree 7 and 11, (2004). (Preprint).

[7] C. U. Jensen, A. Ledet and N. Yui, *Generic polynomials*, Cambridge Univ. Press, Cambridge, 2002.

[8] K. Masuda, On a problem of Chevalley, Nagoya Math. J. **8** (1955), 59–63.

[9] K. Masuda, Application of the theory of the group of classes of projective modules to the existance problem of independent parameters of invariant, J. Math. Soc. Japan **20** (1968), 223–232.

[10] T. Miyata, Invariants of certain groups. I, Nagoya Math. J. **41** (1971), 69–73.

[11] E. Noether, Rationale Funktionenkörper, Jahrbericht Deutsch. Math.-Verein. **22** (1913), 316–319.

[12] E. Noether, Gleichungen mit vorgeschriebener Gruppe, Math. Ann. **78** (1918), 221–229.

[13] A. D. Thomas and G. V. Wood, *Group tables*, Shiva mathematics series, 2, Shiva, Nantwich, 1980.