A conjecture on Euler numbers

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Abstract: In this paper, we will prove that for every prime $p \equiv 1 \pmod 4$, $E_{(p-1)/2} \not\equiv 0 \pmod p$.

Key words: Euler numbers; congruences; class numbers.

1. Introduction. The Euler numbers E_{2n} $(n = 0, 1, 2, \ldots)$ are defined by the Taylor series

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!}, \quad |x| < \frac{\pi}{2}.$$

The following conjecture is on Euler numbers (see [3] B45).

Conjecture 1.1. *If $p \equiv 1 \pmod 4$ is a prime, then $E_{(p-1)/2} \not\equiv 0 \pmod p$.*

Recently, Guodong Liu [2] proved the conjecture for $p \equiv 5 \pmod 8$.

In this paper, using a result of [2] and the class number formula for the quadratic field with negative discriminant, we will prove the above conjecture. We have

Theorem 1.1. *If $p \equiv 1 \pmod 4$ is a prime, then $E_{(p-1)/2} \not\equiv 0 \pmod p$.*

2. Some lemmas. The following Lemma 2.1 due to Liu [2] is crucial to the proof of Theorem 1.1. To be more self-contained, we present a simplified proof here.

Lemma 2.1. *For positive integers n and k, we have*

(1)

$$\sum_{j=0}^{n} \binom{2n}{2j} (2k+1)^{2n-2j} E_{2j} = \sum_{s=0}^{2k} (-1)^s (2k-2s)^{2n}.$$

Proof. For any real number x and any nonnegative integer k, since

$$\left(\sum_{s=0}^{2k} (-1)^s \cos(2k-2s)x \right) \cos x$$
$$= 2 \sum_{s=0}^{k-1} (-1)^s \cos(2k-2s)x \cos x + (-1)^k \cos x$$
$$= \cos(2k+1)x,$$

we have

$$\sum_{s=0}^{2k} (-1)^s \cos(2k-2s)x = \sec x \cdot \cos(2k+1)x, \quad |x| < \frac{\pi}{2}.$$

Thus, we have the following Taylor series

$$\sum_{s=0}^{2k} (-1)^s \sum_{n=0}^{\infty} (-1)^n (2k-2s)^{2n} \frac{x^{2n}}{(2n)!}$$
$$= \left(\sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!} \right) \left(\sum_{n=0}^{\infty} (-1)^n (2k+1)^{2n} \frac{x^{2n}}{(2n)!} \right)$$
$$= \sum_{n=0}^{\infty} (-1)^n \sum_{j=0}^{n} \binom{2n}{2j} (2k+1)^{2n-2j} E_{2j} \frac{x^{2n}}{(2n)!}.$$

It follows that

$$\sum_{j=0}^{n} \binom{2n}{2j} (2k+1)^{2n-2j} E_{2j} = \sum_{s=0}^{2k} (-1)^s (2k-2s)^{2n}.$$

This completes the proof. □

Lemma 2.2 ([1] Corollary 5.3.13.). *If $D < -4$ is a fundamental discriminant, then*

$$h(D) = \frac{1}{D} \sum_{1 \le r < |D|} r \left(\frac{D}{r} \right)$$
$$= \frac{1}{2 - (\frac{D}{2})} \sum_{1 \le r < |D|/2} \left(\frac{D}{r} \right),$$

where (D/r) is the Kronecker symbol (see [1] page 28) and $h(D)$ denotes the class number of the quadratic

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field with discriminant D.

Lemma 2.3. *If* $p \equiv 1 \pmod 4$, *then*

$$h(-4p) = \frac{1}{2}\sum_{s=0}^{p-1}(-1)^s\left(\frac{2s+1}{p}\right) \not\equiv 0 \pmod p.$$

Proof. By Lemma 2.2, we have

$$h(-4p) = \frac{1}{2}\sum_{r=1}^{2p-1}\left(\frac{-4p}{r}\right).$$

Let $r = 2s+1$. Then we have

$$\begin{aligned} h(-4p) &= \frac{1}{2}\sum_{s=0}^{p-1}(-1)^s\left(\frac{-4p}{2s+1}\right) \\ &= \frac{1}{2}\sum_{s=0}^{p-1}(-1)^s\left(\frac{2s+1}{p}\right) < p, \end{aligned}$$

and so $h(-4p) \not\equiv 0 \pmod p$. Lemma 2.3 is proved. □

3. Proof of Theorem 1.1.

Proof of Theorem 1.1. For every positive integers n and k, by Lemma 2.1, we have

$$E_{2n} \equiv \sum_{s=0}^{2k}(-1)^s(2k-2s)^{2n} \pmod{2k+1}. \tag{2}$$

If $p \equiv 1 \pmod 4$ is a prime, we let $k = (p-1)/2$ and $n = (p-1)/4$, then, by (2) and Lemma 2.3, we have

$$\begin{aligned} E_{(p-1)/2} &\equiv \sum_{s=0}^{p-1}(-1)^s(p-2s-1)^{\frac{p-1}{2}} \pmod p \\ &\equiv \sum_{s=0}^{p-1}(-1)^s(2s+1)^{\frac{p-1}{2}} \pmod p \\ &\equiv \sum_{s=0}^{p-1}(-1)^s\left(\frac{2s+1}{p}\right) \pmod p \\ &\equiv 2h(-4p) \not\equiv 0 \pmod p. \end{aligned}$$

This completes the proof. □

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References

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[3] Guy, R. K.: Unsolved Problems in Number Theory. 2nd ed. Springer, New York (1994).