Absolute Frobenius operators

By Nobushige Kurokawa

Department of Mathematics, Tokyo Institute of Technology
Oo-okayama, Meguro-ku, Tokyo 152-8551

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Abstract: We construct the absolute Frobenius operator as a matrix of infinite size for each prime. We calculate the characteristic polynomial and the trace. We investigate the composition of two absolute Frobenius operators also.

Key words: Frobenius automorphism; infinite permutation; infinite matrix; zeta function.

1. Introduction. For a prime number p, let

$$\mathbf{f}_p = \begin{pmatrix} \underbrace{\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}}_{p} &&& \\ & \underbrace{\begin{matrix}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)&&\\&\ddots&\\&&\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\end{matrix}}_{\frac{p^2-p}{2}} && \\ && \underbrace{\begin{matrix}\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right)&&\\&\ddots&\\&&\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right)\end{matrix}}_{\frac{p^3-p}{3}} & \\ &&& \ddots \end{pmatrix}$$

be an infinite matrix, where the number of copies of the size n permutation matrix

$$Z_n = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}$$

is given by

$$\kappa_p(n) = \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right) p^d$$

with the Möbius function $\mu(n)$. This matrix $\mathbf{f}_p$ is obtained as the matrix representation of the Frobenius automorphism Frob_p acting on $\overline{\mathbf{F}_p}$, where $\overline{\mathbf{F}_p}$ denotes the algebraic closure of the finite prime field $\mathbf{F}_p$. Here $\mathrm{Frob}_p(\alpha) = \alpha^p$ for $\alpha \in \overline{\mathbf{F}_p}$, so Frob_p is seen as an infinite permutation on $\overline{\mathbf{F}_p}$. In the words of [KOW], Frob_p acts on the $\mathbf{F}_1$-vector space $\overline{\mathbf{F}}_p$, and the matrix representation $\mathbf{f}_p$ of Frob_p is given via

$$\mathrm{Aut}_{\mathbf{F}_1}(\overline{\mathbf{F}}_p) = GL_\infty(\mathbf{F}_1) = S_\infty \hookrightarrow GL_\infty(\mathbf{Z}).$$

Thus, $\mathbf{f}_p$ respects only the combinatorial structure, so $\mathbf{f}_p$ acts on any countable infinite set. Hence we consider $\mathbf{f}_p$ as the absolute Frobenius operator. We refer to [Kur1, Kur2, M, KK1–KK3, KOW, KW1, KW3, S, Dei] for absolute arithmetic related to this theme; especially we calculated the zeta function associated to the absolute tensor product of several finite prime fields in [Kur1, KK2, KK3, KW1], and we studied $[\partial_p, \partial_q] = \partial_p\partial_q - \partial_q\partial_p$ for absolute derivations ∂_p and ∂_q in [KOW].

In this paper we study basic properties of $\mathbf{f}_p$. First using the calculation of the zeta function of $\mathbf{F}_p[T]$ (or $\mathrm{Spec}\mathbf{F}_p[T]$) due to Kornblum [Kor] we have the characteristic polynomial. For the description we introduce an infinite determinant $\det_\kappa(A)$ as follows: We say that an infinite matrix $A = (a(m,n))_{m,n\geq 1}$ is virtually diagonal if there exists a sequence of finite matrices A_n of size $\kappa(n)$ such that $A = \mathrm{diag}(A_1, A_2, \ldots)$. In this situation, we define

$$\det_\kappa(A) = \prod_{n=1}^{\infty} \det(A_n)$$

when the infinite product converges.

Theorem 1. *For a prime p and $m \geq 1$, we have*

$$\det_{\kappa_p}\left(1 - \mathbf{f}_p^m t\right) = 1 - p^m t.$$

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Theorem 2. *Let*

$$Z(s) = \prod_p \left(1 - \mathbf{f}_p p^{-s}\right)^{-1}$$
$$= 1 + \mathbf{f}_2 2^{-s} + \mathbf{f}_3 3^{-s} + \mathbf{f}_2^2 4^{-s} + \mathbf{f}_5 5^{-s} + \mathbf{f}_2\mathbf{f}_3 6^{-s} + \mathbf{f}_7 7^{-s} + \cdots$$

with the usual ordering on p. *Then*:

(1) $Z(s)$ *converges componentwise absolutely in* $\mathrm{Re}(s) > 1$.

(2) $Z(s) = \begin{pmatrix} \zeta(s) & 0 & \\ 0 & \zeta(s) & 0 \\ & 0 & * \end{pmatrix}$.

(3) $\det Z(s) = \zeta(s-1)$.

Now we look at the trace. We define the trace of an infinite matrix $A = (a(m,n))_{m,n\geq 1}$ as

$$\mathrm{trace}(A) = \sum_{n=1}^{\infty} a(n,n)$$

when this infinite sum converges.

Theorem 3. *For a prime* p *and* $m \geq 1$,

$$\mathrm{trace}(\mathbf{f}_p^m) = p^m.$$

Theorem 4. *Let* p *and* q *be distinct primes. Then*:

(1) $\min\{p,q\} \leq \mathrm{trace}(\mathbf{f}_p\mathbf{f}_q) \leq \max\{p^2,q^2\}$.

(2) $\mathrm{trace}(\mathbf{f}_p\mathbf{f}_q) = \mathrm{trace}(\mathbf{f}_q\mathbf{f}_p)$.

(3) $\mathrm{trace}\left([\mathbf{f}_p,\mathbf{f}_q]\right) = 0$ *for* $[\mathbf{f}_p,\mathbf{f}_q] = \mathbf{f}_p\mathbf{f}_q - \mathbf{f}_q\mathbf{f}_p$.

This shows that the non-commutativity betwen $\mathbf{f}_p$ and $\mathbf{f}_q$ is not detected by the trace. Now we introduce the weighted trace. For a complex number s and an infinite matrix $A = (a(m,n))_{m,n\geq 1}$ we define

$$\mathrm{trace}_s(A) = \sum_{n=1}^{\infty} \frac{a(n,n)}{n^s},$$

which is considered as a zeta function. We remark that formally

$$\mathrm{trace}_{-1}(A) = \sum_{n=1}^{\infty} n \cdot a(n,n)$$

is indicating a Casimir energy (see [KW2, KW3]). For example, in the case of $A = \mathbf{1}$, the infinite unit matrix,

$$\mathrm{trace}_s(\mathbf{1}) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$$

is the Riemann zeta function, and

$$\mathrm{trace}_{-1}(\mathbf{1}) = -\frac{1}{12}$$

is the usual Casimir energy.

Theorem 5. *For distinct primes* p *and* q, $\mathrm{trace}_s([\mathbf{f}_p,\mathbf{f}_q])$ *is non-zero in general. For example*:

(1) $\mathrm{trace}_{-1}([\mathbf{f}_2,\mathbf{f}_3]) = -2$.

(2) $\mathrm{trace}_{-1}([\mathbf{f}_2,\mathbf{f}_5]) = -5$.

(3) $\mathrm{trace}_{-1}([\mathbf{f}_3,\mathbf{f}_5]) = -5$.

This would indicate that the Casimir energy matrix

$$R_{-1} = (\mathrm{trace}_{-1}([\mathbf{f}_p,\mathbf{f}_q]))_{p,q:\ \text{primes}}$$

is an interesting skew symmetric infinite matrix. We hope to return to this theme on another opportunity.

2. Proof of Theorem 1.

From Kornblum [Kor] we have

$$\det{}_{\kappa_p}(1 - \mathbf{f}_p t) = 1 - pt. \tag{2.1}$$

In fact

$$\det{}_{\kappa_p}(1 - \mathbf{f}_p t) = \prod_{n=1}^{\infty}(1-t^n)^{\kappa_p(n)} = 1 - pt, \tag{2.2}$$

where the last equality (2.2) comes from the formula

$$\kappa_p(n) = \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)p^d. \tag{2.3}$$

Actually, from

$$\log\left(\prod_{n=1}^{\infty}(1-t^n)^{\kappa_p(n)}\right) = \sum_{n=1}^{\infty}\kappa_p(n)\log(1-t^n)$$
$$= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\kappa_p(n)}{m}t^{nm}$$
$$= -\sum_{m=1}^{\infty}\frac{1}{m}\left(\sum_{n|m} n\kappa_p(n)\right)t^m$$

and

$$\log(1-pt) = -\sum_{m=1}^{\infty}\frac{1}{m}p^m t^m,$$

we see that the equality

$$\prod_{n=1}^{\infty}(1-t^n)^{\kappa_p(n)} = 1 - pt$$

is equivalent to

$$\sum_{n|m} n\kappa_p(n) = p^m \quad \text{for all} \quad m \geq 1. \tag{2.4}$$

Then, the Möbius inversion formula shows the equivalence between (2.3) and (2.4).

We notice that Dedekind [Ded] obtained the formula (2.3) for $\kappa_p(n)$ as the number of the monic irreducible polynomials in $\mathbf{F}_p[T]$ of degree n. In other words, this means that the space $\overline{\mathbf{F}_p}$ has $\kappa_p(n)$ orbits of length n under the action of Frob_p. This corresponds to

$$\zeta(s, \mathrm{Spec}\mathbf{F}_p[T]) = (1-p^{1-s})^{-1}.$$

Now, we calculate $\det_{\kappa_p}\left(1-\mathbf{f}_p^m t\right)$. A direct way is to notice that Kornblum's formula (2.1) implies

$$\det_{\kappa_p}(1-\mathbf{f}_p\zeta t) = 1-p\zeta t$$

for $\zeta^m = 1$. Then

$$\begin{aligned}\det_{\kappa_p}\left(1-\mathbf{f}_p^m t^m\right) &= \det_{\kappa_p}\left(\prod_{\zeta^m=1}(1-\mathbf{f}_p\zeta t)\right)\\ &= \prod_{\zeta^m=1}\det_{\kappa_p}(1-\mathbf{f}_p\zeta t)\\ &= \prod_{\zeta^m=1}(1-p\zeta t)\\ &= 1-p^m t^m,\end{aligned}$$

and we replace t^m by t. $\square$

3. Proof of Theorem 2.

(1) Let

$$Z(s) = \sum_{n=1}^{\infty}\frac{F(n)}{n^s}.$$

Then, each $F(n)$ is an infinite permutation matrix. Hence matrix components of $F(n)$ are 1 or 0. This shows that $Z(s)$ converges componentwise absolutely in $\mathrm{Re}(s) > 1$.

(2) The matrix components of $F(n)$ at $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$ are calculated in a straightforward way.

(3) Theorem 1 implies

$$\begin{aligned}\det(Z(s)) &= \prod_p \det_{\kappa_p}\left(1-\mathbf{f}_p p^{-s}\right)^{-1}\\ &= \prod_p (1-p^{1-s})^{-1}\\ &= \zeta(s-1).\end{aligned}$$

$\square$

4. Proof of Theorem 3. Notice that

$$\mathrm{trace}(\mathbf{f}_p^m) = \sum_{n=1}^{\infty}\kappa_p(n)\mathrm{trace}(Z_n^m)$$

with

$$\mathrm{trace}(Z_n^m) = \begin{cases} n & \text{if } n|m,\\ 0 & \text{otherwise.}\end{cases}$$

Thus

$$\mathrm{trace}(\mathbf{f}_p^m) = \sum_{n|m} n\kappa_p(n) = p^m,$$

where we used (2.4). $\square$

5. Proof of Theorem 4.

(1) We denote by $\mathbf{f}_p\mathbf{f}_q(n,n)$ the (n,n)-component of the infinite permutation matrix $\mathbf{f}_p\mathbf{f}_q$. We show first that

$$\mathbf{f}_p\mathbf{f}_q(n,n) = 0 \ \text{ if } \ n > \max\{p^2, q^2\}.$$

Suppose that $\mathbf{f}_p\mathbf{f}_q(N,N) = 1$. Denote by $\mathbf{f}_p(m,n)$ and $\mathbf{f}_q(m,n)$ the (m,n)-component of $\mathbf{f}_p$ and $\mathbf{f}_q$ respectively. Then

$$\mathbf{f}_p\mathbf{f}_q(N,N) = \sum_n \mathbf{f}_p(N,n)\mathbf{f}_q(n,N).$$

Write

$$N = \sum_{j=1}^{k-1} j\kappa_p(j) + (l-1)k + m \tag{5.1}$$

with $k \geq 1$, $1 \leq l \leq \kappa_p(k)$, and $1 \leq m \leq k$. This means that the N-th row of $\mathbf{f}_p$ coincides with the m-th row of the l-th matrix Z_k of size k appearing in $\mathbf{f}_p$. Then $\mathbf{f}_p(N,n) = 1$ only for

$$n = \sum_{j=1}^{k-1} j\kappa_p(j) + (l-1)k + (m+1)_k, \tag{5.2}$$

where

$$(m+1)_k = \begin{cases} m+1 & \text{if } m = 1,\dots,k-1\\ 1 & \text{if } m = k.\end{cases}$$

Hence it must be $\mathbf{f}_q(n,N) = 1$ for this n. Writing

$$n = \sum_{j=1}^{k'-1} j\kappa_q(j) + (l'-1)k' + m' \tag{5.3}$$

with $k' \geq 1$, $1 \leq l' \leq \kappa_q(k')$, and $1 \leq m' \leq k'$ similarly to (5.1), we have

$$N = \sum_{j=1}^{k'-1} j\kappa_q(j) + (l'-1)k' + (m'+1)_{k'} \tag{5.4}$$

as in (5.2). Hence, from (5.1)–(5.4) we get

$$m - (m+1)_k = N - n = (m'+1)_{k'} - m'. \tag{5.5}$$

To treat (5.5), we divide into four cases: (a) $m = k$, $m' = k'$, (b) $m = k$, $m' \neq k'$, (c) $m \neq k$, $m' = k'$, (d) $m \neq k$, $m' \neq k'$. In the case (a) we have $k - 1 = 1 - k'$, so we get $k = k' = 1$, hence $N \leq p, q$.

In the case (b) we have $k-1=1$, so $k=2$, and $N \le p^2$. In the case (c) we have $-1=1-k'$, so $k'=2$, and $N \le q^2$. The case (d) does not occur since the condition (5.5) implies $-1=1$ in this case. Thus, we get $N \le \max\{p^2,q^2\}$. So, $\mathbf{f}_p\mathbf{f}_q(n,n)=0$ if $n > \max\{p^2,q^2\}$. Hence

$$\mathrm{trace}(\mathbf{f}_p\mathbf{f}_q) \le \max\{p^2,q^2\}.$$

The inequality

$$\mathrm{trace}(\mathbf{f}_p\mathbf{f}_q) \ge \min\{p,q\}$$

is obvious.

(2) Here, it is convenient to use the permutational description. We denote by $f_p \in S_\infty$ the permutation corresponding to $\mathbf{f}_p$. In other words $\mathbf{f}_p$ is the matrix representation of f_p:

$$\mathbf{f}_p = \left(\delta_{m f_p(n)}\right)_{m,n\ge 1}.$$

Then we get

$$\mathrm{trace}(\mathbf{f}_p\mathbf{f}_q) = \#\mathrm{Fix}(f_pf_q)$$

and

$$\mathrm{trace}(\mathbf{f}_q\mathbf{f}_p) = \#\mathrm{Fix}(f_qf_p),$$

where

$$\mathrm{Fix}(\sigma) = \{n=1,2,3,\dots \mid \sigma(n)=n\}$$

for $\sigma \in S_\infty$. Hence it is sufficient to show

$$\#\mathrm{Fix}(f_pf_q) = \#\mathrm{Fix}(f_qf_p),$$

and we see this equality from the bijection $\mathrm{Fix}(f_pf_q) \to \mathrm{Fix}(f_qf_p)$ given by $n \mapsto f_p^{-1}(n)$.

(3) This follows from (2). □

6. Proof of Theorem 5. The results are obtained from direct calculations as follows:

(1)

$$\begin{aligned}
\mathrm{trace}_s\,(\mathbf{f}_2\mathbf{f}_3) &= 1+2^{-s}+6^{-s}+8^{-s},\\
\mathrm{trace}_s\,(\mathbf{f}_3\mathbf{f}_2) &= 1+2^{-s}+7^{-s}+9^{-s},\\
\mathrm{trace}_s\,([\mathbf{f}_2,\mathbf{f}_3]) &= 6^{-s}-7^{-s}+8^{-s}-9^{-s},
\end{aligned}$$

(2)

$$\begin{aligned}
\mathrm{trace}_s\,(\mathbf{f}_2\mathbf{f}_5) &= 1+2^{-s}+6^{-s}+12^{-s}+16^{-s}\\
&\quad +20^{-s}+24^{-s},\\
\mathrm{trace}_s\,(\mathbf{f}_5\mathbf{f}_2) &= 1+2^{-s}+7^{-s}+13^{-s}+17^{-s}\\
&\quad +21^{-s}+25^{-s},\\
\mathrm{trace}_s\,([\mathbf{f}_2,\mathbf{f}_5]) &= 6^{-s}-7^{-s}+12^{-s}-13^{-s}\\
&\quad +16^{-s}-17^{-s}+20^{-s}-21^{-s}\\
&\quad +24^{-s}-25^{-s},
\end{aligned}$$

(3)

$$\begin{aligned}
\mathrm{trace}_s\,(\mathbf{f}_3\mathbf{f}_5) &= 1+2^{-s}+3^{-s}+6^{-s}+7^{-s}\\
&\quad +8^{-s}+9^{-s}+10^{-s}+14^{-s}\\
&\quad +16^{-s}+20^{-s}+22^{-s},\\
\mathrm{trace}_s\,(\mathbf{f}_5\mathbf{f}_3) &= 1+2^{-s}+3^{-s}+6^{-s}+7^{-s}\\
&\quad +8^{-s}+9^{-s}+11^{-s}+15^{-s}\\
&\quad +17^{-s}+21^{-s}+23^{-s},\\
\mathrm{trace}_s\,([\mathbf{f}_3,\mathbf{f}_5]) &= 10^{-s}-11^{-s}+14^{-s}-15^{-s}\\
&\quad +16^{-s}-17^{-s}+20^{-s}-21^{-s}\\
&\quad +22^{-s}-23^{-s}.
\end{aligned}$$

We notice that the permutational description f_p is more comfortable for these calculations. □

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