On a holomorphic curve extremal for the defect relation

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Abstract: Let f be a transcendental holomorphic curve from the complex plane into the two dimensional complex projective space of which defect relation over a set X in N-subgeneral position is extremal. Then, there are $N-1$ vectors in X whose deficiency with respect to f is equal to 1.

Key words: Holomorphic curve; defect relation; extremal; subgeneral position.

1. Introduction. Let $f=[f_1,\dots,f_{n+1}]$ be a holomorphic curve from $\boldsymbol{C}$ into the n-dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation

$$(f_1,\dots,f_{n+1}):\boldsymbol{C}\to\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\},$$

where n is a positive integer. We use the following notations:

$$\|f(z)\|=(|f_1(z)|^2+\cdots+|f_{n+1}(z)|^2)^{1/2}$$

and for a vector $\boldsymbol{a}=(a_1,\dots,a_{n+1})\in\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$

$$\|\boldsymbol{a}\|=(|a_1|^2+\cdots+|a_{n+1}|^2)^{1/2},$$
$$(\boldsymbol{a},f(z))=a_1f_1(z)+\cdots+a_{n+1}f_{n+1}(z),$$
$$(\boldsymbol{a},f)=a_1f_1+\cdots+a_{n+1}f_{n+1}.$$

The characteristic function $T(r,f)$ of f is defined as follows (see [10]):

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{i\theta})\|d\theta-\log\|f(0)\|.$$

We suppose throughout the paper that f is transcendental: $\lim_{r\to\infty}T(r,f)/\log r=\infty$ and that f is linearly non-degenerate over $\boldsymbol{C}$; namely, $f_1,\dots,f_{n+1}$ are linearly independent over $\boldsymbol{C}$.

For meromorphic functions in the complex plane we use the standard notations of the Nevanlinna theory of meromorphic functions ([4, 5]).

For $\boldsymbol{a}\in\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$, we write

$$m(r,\boldsymbol{a},f)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|\boldsymbol{a}\|\|f(re^{i\theta})\|}{|(\boldsymbol{a},f(re^{i\theta}))|}d\theta,$$
$$N(r,\boldsymbol{a},f)=N(r,1/(\boldsymbol{a},f)).$$

We call the quantity

$$\delta(\boldsymbol{a},f)=1-\limsup_{r\to\infty}\frac{N(r,\boldsymbol{a},f)}{T(r,f)}=\liminf_{r\to\infty}\frac{m(r,\boldsymbol{a},f)}{T(r,f)}$$

the deficiency of $\boldsymbol{a}$ with respect to f. It is known that $0\le\delta(\boldsymbol{a},f)\le 1$.

Let X be a subset of $\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$ in N-subgeneral position; that is to say,

(i) $\#X\ge 2N-n+2$ and

(ii) any $N+1$ elements of X generate $\boldsymbol{C}^{n+1}$,

where N is an integer satisfying $N\ge n$.

Cartan ([1], $N=n$) and Nochka ([6], $N>n$) gave the following

Theorem A (Defect Relation). *For any q elements $\boldsymbol{a}_1,\dots,\boldsymbol{a}_q$ of X*

$$\sum_{j=1}^{q}\delta(\boldsymbol{a}_j,f)\le 2N-n+1$$

($2N-n+1\le q\le\infty$) (*see also* [2] *or* [3]).

We are interested in the holomorphic curve f extremal for the defect relation:

$$\sum_{j=1}^{q}\delta(\boldsymbol{a}_j,f)=2N-n+1. \tag{1}$$

In [9] we proved the following theorem.

Theorem B. *Suppose that there are vectors $\boldsymbol{a}_1,\dots,\boldsymbol{a}_q$ in X such that* (1) *holds, where $2N-n+1<q\le\infty$. If $N>n$ and n is even, then there are at least $[(2N-n+1)/(n+1)]+1$ vectors $\boldsymbol{a}\in\{\boldsymbol{a}_1,\dots,\boldsymbol{a}_q\}$ satisfying $\delta(\boldsymbol{a},f)=1$.*

The purpose of this paper is to improve Theorem B when $n=2$:

Theorem. *Suppose that $N>n=2$ and that there are vectors $\boldsymbol{a}_1,\dots,\boldsymbol{a}_q$ in X such that* (1) *holds,*

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where $2N-1<q\leq\infty$*. Then there are at least* $N-1$ *vectors* $\boldsymbol{a}\in\{\boldsymbol{a}_1,\dots,\boldsymbol{a}_q\}$ *satisfying* $\delta(\boldsymbol{a},f)=1$.

2. Preliminaries. We shall give some lemmas in this section for later use. Let $f=[f_1,\dots,f_{n+1}]$, X etc. be as in Section 1, q any integer satisfying $2N-n+1<q<\infty$ and we put $Q=\{1,2,\dots,q\}$.

Let $\{\boldsymbol{a}_j\mid j\in Q\}$ be a family of vectors in X. For a non-empty subset P of Q, we denote

$V(P)=$ the vector space spanned by $\{\boldsymbol{a}_j\mid j\in P\}$,

$d(P)=\dim V(P)$

and we put $\mathcal{O}=\{P\subset Q\mid 0<\#P\leq N+1\}$.

Lemma 2.1 ((2.4.3) in [3, p. 68]). *For* $P\in\mathcal{O}$, $\#P-d(P)\leq N-n$.

Lemma 2.2 ([9, Proposition 10 (II)]). *Suppose that there exists a function* $\tau:Q\to(0,1]$ *which satisfies the following condition* $(*)$:

$(*)$ *For any* $P\in\mathcal{O}$, $\sum_{j\in P}\tau(j)\leq d(P)$.

Then, for vectors $\boldsymbol{a}_1,\dots,\boldsymbol{a}_q\in X$*, we have the inequality*:

$$\sum_{j=1}^{q}\tau(j)\delta(\boldsymbol{a}_j,f)\leq n+1.$$

3. Theorem. From now on throughout this paper we suppose that $N>n=2$. Then, the holomorphic curve $f=[f_1,f_2,f_3]$ is transcendental from $\boldsymbol{C}$ into the two dimensional projective space $P^2(\boldsymbol{C})$, X is a subset of $\boldsymbol{C}^3-\{\boldsymbol{0}\}$ in N-subgeneral position.

From Theorem A it is easy to see that the set $\{\boldsymbol{a}\in X\mid\delta(\boldsymbol{a},f)>0\}$ is at most countable and

$$\sum_{\boldsymbol{a}\in X}\delta(\boldsymbol{a},f)\leq 2N-1.$$

We call this inequality the defect relation of f over X.

(**A**) First we consider the extremal holomorphic curve f with a finite number of vectors $\boldsymbol{a}\in X$ satisfying $\delta(\boldsymbol{a},f)>0$.

Suppose that there are vectors $\boldsymbol{a}_1,\dots,\boldsymbol{a}_q$ in X satsfying

$$\sum_{j=1}^{q}\delta(\boldsymbol{a}_j,f)=2N-1, \tag{2}$$

where $2N-1<q<\infty$.

As in Section 2, we put $Q=\{1,2,\dots,q\}$, for a non-empty subset P of Q, $V(P)$ is the vector space spanned by $\{\boldsymbol{a}_j\mid j\in P\}$, $d(P)=\dim V(P)$ and we put $\mathcal{O}=\{P\subset Q\mid 0<\#P\leq N+1\}$.

Definition 3.1 ([8, Definition 1]). We put

$$\lambda=\min_{P\in\mathcal{O}}d(P)/\#P.$$

Proposition 3.1. $1/(N-1)\leq\lambda<3/(2N-1)$.

In fact, we obtain the first inequality from [8, Proposition 2] for $n=2$ and the second one from [9, p. 295] for $n=2$.

Let P_0 be an element of $\mathcal{O}$ satisfying $d(P_0)/\#P_0=\lambda$. Then, we have the following

Proposition 3.2. $d(P_0)=1$ *and* $\#P_0\leq N-1$.

Proof. As $P_0\neq\phi$, $d(P_0)\geq 1$. By Proposition 3.1 and Lemma 2.1, we have the inequality

$$d(P_0)<\frac{3}{2N-1}\#P_0\leq\frac{3}{2N-1}(N-2+d(P_0)),$$

so that we have $d(P_0)<3/2$, which means that $d(P_0)=1$. This implies that $\#P_0\leq N-1$ by Lemma 2.1. □

Proposition 3.3 ([9, Remark 1, Theorem 1]). *For* $j\in P_0$, $\delta(\boldsymbol{a}_j,f)=1$.

To prove our theorem when q is finite, we have only to prove that $\#P_0=N-1$ by Proposition 3.3. Let $\#P_0=N-x+1$. Then, $x\geq 2$ by Proposition 3.2 and

$$\lambda=\min_{P\in\mathcal{O}}\frac{d(P)}{\#P}=\frac{d(P_0)}{\#P_0}=\frac{1}{N-x+1}.$$

Proposition 3.4. $x<(N+4)/3$.

Proof. As $\lambda<3/(2N-1)$ by Proposition 3.1, we have the inequality

$$1/(N-x+1)<3/(2N-1),$$

which reduces to $x<(N+4)/3$. □

Proposition 3.5. *Let* $P\in\mathcal{O}$*. If* $P\subset Q\setminus P_0$ *and* $\#P\geq x$*, then* $d(P)\geq 2$ *and* $d(P\cup P_0)=3$.

Proof. As X is in N-subgeneral position and $\#(P_0\cup P)\geq N+1$, we have that $d(P_0\cup P)=3$. Further as $d(P_0)=1$, we have that $d(P)\geq 2$. □

Proposition 3.6. *Let* $P\in\mathcal{O}$*. If* $P\setminus P_0\neq\phi$ *and* $P\cap P_0\neq\phi$*, then* $d(P)/\#P\geq 2/N$.

Proof. First, we prove that $d(P)\geq 2$. Suppose to the contrary that $d(P)=1$. Then $d(P_0\cup P)=1$ because $P\cap P_0\neq\phi$ and $d(P_0)=1$. Further, we have that

$$N-x+1<\#(P_0\cup P)\leq N-1$$

since $\#P_0 = N - x + 1$, $P \setminus P_0 \neq \phi$, $d(P_0 \cup P) = 1$ and X is in N-subgeneral positon. We then have that $P_0 \cup P \in \mathcal{O}$ and

$$d(P_0 \cup P)/\#(P_0 \cup P) < 1/(N - x + 1) = \lambda,$$

which contradicts the definition of λ. We have that $d(P) \geq 2$.

When $d(P) = 2$, we have that $\#P \leq N$ and $d(P)/\#P \geq 2/N$.

When $d(P) = 3$, $d(P)/\#P \geq 3/(N+1) > 2/N$ since $\#P \leq N + 1$. □

Proposition 3.7. *Let $P \in \mathcal{O}$. If $P \cap P_0 = \phi$, then $d(P)/\#P \geq 2/N$.*

Proof. (i) When $d(P) = 1$, $\#P \leq x - 1$ by Proposition 3.5 since if $\#P \geq x$, then $d(P) \geq 2$. In this case we have the inequality

$$\frac{d(P)}{\#P} = \frac{1}{\#P} \geq \frac{1}{x-1} > \frac{3}{N+1}$$

by Proposition 3.4.

(ii) When $d(P) \geq 2$, we have the inequality $d(P)/\#P \geq 2/N$ as in Proposition 3.6.

As $3/(N+1) > 2/N$, we have this proposition from (i) and (ii). □

Remark 3.1. We note that $P \setminus P_0 \neq \phi$ in Proposition 3.7.

Definition 3.2. We put

$$\mathcal{O}_1 = \{P \in \mathcal{O} \mid P \setminus P_0 \neq \phi\} \quad \text{and} \quad \lambda_1 = \min_{P \in \mathcal{O}_1} \frac{d(P)}{\#P}.$$

Remark 3.2. $\lambda_1 \geq 2/N$ by Propositions 3.6 and 3.7.

Proposition 3.8. $\lambda < \lambda_1$.

Proof. By Remark 3.2, we have the inequality

$$\lambda_1 - \lambda \geq \frac{2}{N} - \frac{1}{N-x+1} = \frac{N-2x+2}{N(N-x+1)} > 0$$

since $N \geq 3$ and $(N+2)/2 > (N+4)/3 > x$ by Proposition 3.4. □

Definition 3.3. We put

$$\sigma(j) = \begin{cases} \lambda & (j \in P_0) \\ \lambda_1 & (j \in Q \setminus P_0). \end{cases}$$

Note that $0 < \sigma(j) \leq 1$ $(j \in Q)$ from Definitions 3.1, 3.2 and 3.3.

Proposition 3.9. *For any $P \in \mathcal{O}$, we have the inequality $\sum_{j \in P} \sigma(j) \leq d(P)$.*

Proof. (i) When $P \subset P_0$,

$$\sum_{j \in P} \sigma(j) = \lambda \#P \leq \frac{d(P)}{\#P} \#P = d(P).$$

(ii) When $P \setminus P_0 \neq \phi$,

$$\sum_{j \in P} \sigma(j) \leq \lambda_1 \#P \leq \frac{d(P)}{\#P} \#P = d(P).$$

□

Proposition 3.10. $\sum_{j=1}^{q} \sigma(j)\delta(\boldsymbol{a}_j, f) \leq 3$.

Proof. We obtain this inequality from Proposition 3.9 and Lemma 2.2 for $n = 2$. □

Proposition 3.11. $\#P_0 = N - 1$.

Proof. From Proposition 3.10 we have the inequality

$$(3) \qquad \sum_{j \in P_0} \sigma(j)\delta(\boldsymbol{a}_j, f) + \sum_{j \in Q \setminus P_0} \sigma(j)\delta(\boldsymbol{a}_j, f) \leq 3.$$

As $\delta(\boldsymbol{a}_j, f) = 1$ $(j \in P_0)$ (Proposition 3.3), from (3) we have the inequality

$$\frac{1}{N-x+1}(N-x+1) + \sum_{j \in Q \setminus P_0} \sigma(j)\delta(\boldsymbol{a}_j, f) \leq 3.$$

As $\sigma(j) = \lambda_1 \geq 2/N$ $(j \in Q \setminus P_0)$ (Remark 3.2), we have the inequality

$$\frac{2}{N} \sum_{j \in Q \setminus P_0} \delta(\boldsymbol{a}_j, f) \leq 2, \quad \text{or} \quad \sum_{j \in Q \setminus P_0} \delta(\boldsymbol{a}_j, f) \leq N.$$

On the other hand, from (2) we have the equality

$$\begin{aligned} \sum_{j \in Q \setminus P_0} \delta(\boldsymbol{a}_j, f) &= 2N - 1 - (N - x + 1) \\ &= N + x - 2, \end{aligned}$$

so that we have $N + x - 2 \leq N$ or $x \leq 2$, which means that $\#P_0 = N - x + 1 \geq N - 1$.

Combining this with Proposition 3.2, we have that $\#P_0 = N - 1$. □

(**B**) Next, we consider the extremal holomorphic curve f with an infinite number of vectors $\boldsymbol{a}_j \in X$ such that $\delta(\boldsymbol{a}_j, f) > 0$ and

$$\sum_{j=1}^{\infty} \delta(\boldsymbol{a}_j, f) = 2N - 1.$$

Let

$$\begin{aligned} &\boldsymbol{N} = \{1, 2, 3, \ldots\} \text{ (the set of positive integers)}, \\ &Y = \{\boldsymbol{a}_j \mid j \in \boldsymbol{N}\}, \\ &\mathcal{O}_\infty = \{P \subset \boldsymbol{N} \mid 0 < \#P < N + 1\} \end{aligned}$$

and for any subset P of $\boldsymbol{N}$, we use the notations $V(P)$ and $d(P)$ as in Section 2.

Definition 3.4 ([8, p. 144])**.** We put

$$\mu = \min_{P \in \mathcal{O}_\infty} d(P)/\#P.$$

Note that the set $\{d(P)/\#P \mid P \in \mathcal{O}_\infty\}$ is a finite set.

Proposition 3.1′. $1/(N-1) \le \mu < 3/(2N-1)$.

In fact, we have the first inequality from [8, p. 144] for $n = 2$ and the second one from [9, pp. 298–299] for $n = 2$.

Let P_0 be an element of $\mathcal{O}_\infty$ satisfying $\mu = d(P_0)/\#P_0$. As in the case of Proposition 3.2, we have the following

Proposition 3.2′. $d(P_0) = 1$ *and* $\#P_0 \le N-1$.

Further we have the following

Proposition 3.3′ ([9, Proof of Theorem 2, pp. 299–300])**.** *For* $j \in P_0$, $\delta(\boldsymbol{a}_j, f) = 1$.

To prove our theorm when q is infinite, we have only to prove that $\#P_0 = N-1$ by Proposition 3.3′. Let $\#P_0 = N-x+1$. Then, $x \ge 2$ by Proposition 3.2′ and

$$\mu = \min_{P \in \mathcal{O}_\infty} \frac{d(P)}{\#P} = \frac{d(P_0)}{\#P_0} = \frac{1}{N-x+1}.$$

Remark 3.3. As in the case (**A**), we obtain the same propositions as in Propositions 3.4, 3.5, 3.6 and 3.7 for P_0 in this case.

For any positive number $0 < \epsilon < 1$, we choose an integer q satisfying $Q = \{1, 2, \ldots, q\} \supset P_0$, $q > 2N-1$ and

$$2N - 1 - \epsilon < \sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f). \tag{4}$$

We put $\mathcal{P} = \{P \subset Q \mid 0 < \#P \le N+1\}$.

Note that $\mu = \min_{P \in \mathcal{P}} d(P)/\#P$ since $\mathcal{P} \ni P_0$ and $\mu = d(P_0)/\#P_0$.

Definition 3.5. We put

$$\mathcal{P}_1 = \{P \in \mathcal{P} \mid P \setminus P_0 \ne \phi\} \text{ and } \mu_1 = \min_{P \in \mathcal{P}_1} \frac{d(P)}{\#P}.$$

Note that $\mathcal{P}_1 \ne \phi$ since $\#Q > 2N-1$ and $\#P_0 \le N-1$.

Remark 3.4. $\mu_1 \ge 2/N$ as in Remark 3.2.

As in the case of Proposition 3.8, we have the following

Proposition 3.8′. $\mu < \mu_1$.

Definition 3.6. We put

$$\tau(j) = \begin{cases} \mu & (j \in P_0) \\ \mu_1 & (j \in Q \setminus P_0). \end{cases}$$

From Definitions 3.4, 3.5 and 3.6, we have that $\tau : Q \to (0, 1]$. As in the case of Proposition 3.9, we have the following

Proposition 3.9′. *For any* $P \in \mathcal{P}$, *we have the inequality* $\sum_{j \in P} \tau(j) \le d(P)$.

By using this proposition, we have the following proposition as in Proposition 3.10:

Proposition 3.10′. $\sum_{j=1}^{q} \tau(j)\delta(\boldsymbol{a}_j, f) \le 3$.

Finally, we obtain the following proposition corresponding to Proposition 3.11.

Proposition 3.11′. $\#P_0 = N-1$.

Proof. From Proposition 3.10′ we have the inequality

$$\sum_{j \in P_0} \tau(j)\delta(\boldsymbol{a}_j, f) + \sum_{j \in Q \setminus P_0} \tau(j)\delta(\boldsymbol{a}_j, f) \le 3. \tag{5}$$

As $\delta(\boldsymbol{a}_j, f) = 1$ $(j \in P_0)$ (Proposition 3.3′), from (5) we have the inequality

$$\frac{1}{N-x+1}(N-x+1) + \sum_{j \in Q \setminus P_0} \tau(j)\delta(\boldsymbol{a}_j, f) \le 3.$$

As $\tau(j) = \mu_1 \ge 2/N$ $(j \in Q \setminus P_0)$ (Remark 3.4), we have the inequality

$$\frac{2}{N} \sum_{j \in Q \setminus P_0} \delta(\boldsymbol{a}_j, f) \le 2, \quad \text{or} \quad \sum_{j \in Q \setminus P_0} \delta(\boldsymbol{a}_j, f) \le N.$$

On the other hand, from (4) we have the inequality

$$\begin{aligned} \sum_{j \in Q \setminus P_0} \delta(\boldsymbol{a}_j, f) &> 2N - 1 - \epsilon - (N - x + 1) \\ &= N + x - 2 - \epsilon, \end{aligned}$$

so that we have $N + x - 2 - \epsilon < N$ or $x \le 2 + \epsilon$. This means that $\#P_0 = N - x + 1 \ge N - 1 - \epsilon$, and so we have that $\#P_0 \ge N-1$ as P_0 is an integer and $0 < \epsilon < 1$. Combining this with Proposition 3.2′ we have this proposition. □

Summarizing the results obtained in this section we have our Theorem:

Theorem 3.1. *Suppose that* $N > 2$ *and that there are vectors* $\boldsymbol{a}_j$ $(j = 1, \ldots, q) \in X$ $(2N-1 < q \le \infty)$ *satisfying*

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) = 2N - 1.$$

Then, there exists a subset P_0 *of* $\{1, 2, \ldots, q\}$ *such that*

(i) $d(P_0) = 1$ *and* $\#P_0 = N-1$;

(ii) $\delta(\boldsymbol{a}_j, f) = 1$ *for* $j \in P_0$.

4. Example. Let f, X and $N > n = 2$ be as in Section 3. Theorem 3.1 implies that for f to be extremal for the defect relation it is necessary that there exists a subset S_0 of X satisfying

$$\#S_0 = N - 1 \quad \text{and} \quad d(S_0) = 1, \tag{6}$$

where $d(S_0)$ is the dimension of the vector space spanned by the elements of S_0.

This shows that if X does not have any subset satisfying (6), any transcendental holomorphic curve is not extremal for the defect relation over X. In this section, we shall give an example of f and X which satisfy Theorem 3.1 and an example of *maximal* subset of $\boldsymbol{C}^3 - \{\boldsymbol{0}\}$ in N-subgeneral position having no subset satisfying (6). We use $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ as the standard basis of $\boldsymbol{C}^3$.

Example 4.1. Let $f_1 = [e^z, z, 1]$. For $N > 2$ we put

$$X_1 = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{2N-1}\} \cup \{(a^2, a, 1) \mid a \in \boldsymbol{C}, a \neq 0, 1, \ldots, N-2\},$$

where

$$\boldsymbol{a}_j = j\boldsymbol{e}_1 \ (1 \leq j \leq N-1); \quad \boldsymbol{a}_{N+k} = k\boldsymbol{e}_2 + \boldsymbol{e}_3 \ (0 \leq k \leq N-2); \quad \boldsymbol{a}_{2N-1} = \boldsymbol{e}_2.$$

Then, f_1 is transcendental; X_1 is in N-subgeneral position and the defect relation of f_1 over X_1 is extremal.

Proof. The characteristic function $T(r, f_1)$ satisfies the inequality

$$r/\pi + O(1) \leq T(r, f_1) \leq r/\pi + \log r + O(1) \tag{7}$$

by [7, Lemme 1] and [4, pp. 6–7]. This implies that f_1 is transcendental. By the definition we have the estimates

$$N(r, \boldsymbol{a}_j, f_1) = \begin{cases} 0 & (j = 1, \ldots, N); \\ \log r + O(1) & (j = N+1, \ldots, 2N-1), \end{cases}$$

and so from (7) we obtain that

$$\delta(\boldsymbol{a}_j, f_1) = 1 \ (j = 1, \ldots, 2N-1).$$

It is easy to see that X_1 is in N-subgeneral position, and so by Theorem A $\delta(\boldsymbol{a}, f_1) = 0$ for $\boldsymbol{a} \in X_1 - \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{2N-1}\}$ and we have the equality

$$\sum_{\boldsymbol{a} \in X_1} \delta(\boldsymbol{a}, f) = 2N - 1.$$

□

Definition 4.1. We say that X is maximal if for any W in N-subgeneral position such that

$$X \subset W \subset \boldsymbol{C}^3 - \{\boldsymbol{0}\}, \quad \text{then} \quad W = X.$$

We consider the following subset X_2 of $\boldsymbol{C}^3 - \{\boldsymbol{0}\}$.

Example 4.2. We put

$$X_2 = \{j\boldsymbol{e}_1 \mid j = 1, \ldots, N-2\} \cup \{\boldsymbol{e}_2, 2\boldsymbol{e}_2\} \cup \{k(a^2, a, 1) \mid a \in \boldsymbol{C}; k = 1, 2\}.$$

Proposition 4.1. *If $N \geq 6$, X_2 is in N-subgeneral position.*

Proof. Let S be any subset of X_2 such that $\#S = N + 1$. We have only to prove that there are three elements in S which are linearly independent.

(a) The case when S contains at least one $j_1\boldsymbol{e}_1$ $(1 \leq j_1 \leq N-2)$ and $\alpha\boldsymbol{e}_2$ $(\alpha = 1$ or $2)$.

S must contain a vector $k(a^2, a, 1)$ $(k = 1$ or 2; $a \in \boldsymbol{C})$. Then it is easy to see that three vectors $j_1\boldsymbol{e}_1$, $\alpha\boldsymbol{e}_2$ and $k(a^2, a, 1)$ are linearly independent.

(b) The case when S contains $j_1\boldsymbol{e}_1$ $(1 \leq j_1 \leq N-2)$, but does not contain $\alpha\boldsymbol{e}_2$ $(\alpha = 1, 2)$.

S must contain two vectors

$$k_1(a_1^2, a_1, 1), \quad k_2(a_2^2, a_2, 1) \quad (k_1, k_2 = 1 \ \text{or} \ 2; a_1 \neq a_2 \in \boldsymbol{C}).$$

Then, three vectors $j_1\boldsymbol{e}_1$, $k_1(a_1^2, a_1, 1)$, $k_2(a_2^2, a_2, 1)$ are linearly independent.

(c) The case when S does not contain any one of $\{j\boldsymbol{e}_1 \mid j = 1, \ldots, N-2\}$.

As $N \geq 6$, S must contain the following three vectors:

$$k_1(a_1^2, a_1, 1), \quad k_2(a_2^2, a_2, 1), \quad k_3(a_3^2, a_3, 1),$$

where $k_1, k_2, k_3 = 1$ or 2 and a_1, a_2 and a_3 are distinct complex numbers. Then, these three vectors are linearly independent.

From (a), (b) and (c), S contains three independent vectors. This means that X_2 is in N-subgeneral position. □

Remark 4.1. It is easy to see that X_2 is not in $N-1$ subgeneral position as N vectors $\{j\boldsymbol{e}_1 \mid j = 1, \ldots, N-2\} \cup \{\boldsymbol{e}_2, 2\boldsymbol{e}_2\}$ do not contain three independent vectors.

Proposition 4.2. *If $N \geq 6$, X_2 is maximal.*

Proof. We have only to prove that for any vector $(\alpha, \beta, \gamma) \in \boldsymbol{C}^3 - \{\boldsymbol{0}\}$ not belonging to X_2, the set $X_2 \cup \{(\alpha, \beta, \gamma)\}$ is not in N-subgeneral position.

(a) The case when $\gamma = 0$. It is easy to see that $N + 1$ vectors

$$e_1, 2e_1, \ldots, (N-2)e_1, e_2, 2e_2, (\alpha, \beta, 0)$$

do not contain three independent vectors.

(b) The case when $\gamma \neq 0$. Put $\beta/\gamma = a$. Then, it is easy to see that $N+1$ vectors

$$e_1, 2e_1, \ldots, (N-2)e_1, (a^2, a, 1), 2(a^2, a, 1), (\alpha, \beta, \gamma)$$

do not contain three independent vectors.

From (a) and (b) we have that $X_2 \cup \{(\alpha, \beta, \gamma)\}$ is not in N-subgeneral position. □

Theorem 4.1. *If $N \geq 6$, for any transcendental holomorphic curve f from $\boldsymbol{C}$ into $P^2(\boldsymbol{C})$, the defect relation of f over X_2 is not extremal.*

Proof. Suppose that there exists a transcendental holomorphic curve f from $\boldsymbol{C}$ into $P^2(\boldsymbol{C})$ satisfying

$$\sum_{\boldsymbol{a} \in X_2} \delta(\boldsymbol{a}, f) = 2N - 1.$$

Then, by Theorem 3.1, there must exist $N-1$ vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1}$ in X_2 such that

(i) the vector space spanned by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1}$ is of dimension 1 and

(ii) $\delta(\boldsymbol{a}_j, f) = 1$ $(j = 1, \ldots, N-1)$.

But, X_2 does not contain $N-1$ vectors satisfying (i). This is a contradiction. We have our theorem. □

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