Computable sequences in the Sobolev spaces

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Abstract: Pour-El and Richards [5] discussed computable smooth functions with non-computable first derivatives. We show that a similar result holds in the case of Sobolev spaces by giving a non-computable $\mathcal{H}^1(0,1)$-element which, however, is computable in any of larger Sobolev spaces $\mathcal{H}^s(0,1)$ for any computable s, $0 \le s < 1$.

Key words: Effective and non-effective convergence; Sobolev spaces.

1. Introduction. Let Ω be an open set in a d-dimensional Euclidean space $\mathbf{R}^d$. The Sobolev space $\mathcal{H}^m(\Omega)$ of order m, $(m = 0, 1, 2, \cdots)$, over Ω is a Hilbert space consisting of the Lebesgue measurable (complex valued) functions $u(x)$ such that it and all of its weak derivatives up to order m inclusive are square summable over Ω. The inner-product of $\mathcal{H}^m(\Omega)$ is given by

$$(u\,,\,v)_m = \sum_{|\alpha|\le m} \int_\Omega \partial^\alpha\, u(x) \cdot \overline{\partial^\alpha\, v(x)}\, dx,$$

for u, $v \in \mathcal{H}^m(\Omega)$. Here $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbf{N}^n$ are multi-indices. Thus, the length of α is $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Recall also $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ for a partial derivation of order α. Recall $\|u\|_m = \sqrt{(u,\,u)_m}$ defines the norm of $u \in \mathcal{H}^m(\Omega)$. In particular, the Sobolev space of order 0, $\mathcal{H}^0(\Omega)$, coincides with the Lebesgue space $\mathcal{L}^2(\Omega)$ of the square summable functions. For these function spaces, see any standard textbook of partial differential equations or functional analysis. See, e.g., Adams [1], also Hörmander [4]. The computability notion in a separable Hilbert space is discussed in Pour-El and Richards [5]. Computablity properties of the Sobolev spaces are discussed in Zhong [7] for the case $\Omega = \mathbf{R}^d$.

The inclusion relation

$$\mathcal{H}^m(\Omega) \subset \mathcal{H}^l(\Omega),\ m > l \ge 0, \tag{1}$$

is clear from the definition. (1) means that the canonical injection

$$\mathcal{H}^m(\Omega) \ni u \mapsto u \in \mathcal{H}^l(\Omega), \quad m > l \ge 0 \tag{2}$$

is continuous.

It is a classical fact that if Ω has a *nice* ($\mathcal{C}^\infty$) boundary $\partial\Omega$, then $\bigcap_{\ell=0}^\infty \mathcal{H}^l(\Omega)(\subset \mathcal{C}^\infty(\Omega))$ is dense in each of $\mathcal{H}^m(\Omega)$. Actually, the set spanned by the $\mathcal{C}^\infty$ functions supported in closed disks intersecting with Ω, centered at rational points and with rational radii, is contained in $\bigcap_{\ell=0}^\infty \mathcal{H}^l(\Omega)$ and dense in each $\mathcal{H}^m(\Omega)$. Note then that we have a common effective generating set for all the $\mathcal{H}^m(\Omega)$ consisting of rational dilations and translations of a fixed $\mathcal{C}^\infty$ function supported in the unit disk (as the one analogous to $\varphi(t)$ given below). Thus, by the First Main Theorem of Pour-El and Richards [5], the injection (2) preserves computability. In particular, in the present context, if u is computable in $\mathcal{H}^m(\Omega)$, then so is it in $\mathcal{H}^\ell(\Omega)$, $m > \ell \ge 0$.

However, the mapping (2) also maps non-computable elements in smaller spaces $\mathcal{H}^m(\Omega)$ to computable elements in larger spaces $\mathcal{H}^\ell(\Omega)$. Similar phenomena have been observed for computability in the standard sense of Turing/Lacombe/Grzegorczyk: There is a computable function f (on the real line $\mathbf{R}$), which is continuously differentiable, but with non-computable derivative f' (See [5]).

Modifying the related arguments in [5], we get, in fact, an example of a computable sequence of elements which is non-effectively convergent in $\mathcal{H}^m(\Omega)$, in both of the weak and strong topologies, and which, nevertheless, converges effectively in any of larger spaces $\mathcal{H}^l(\Omega)$, $m > l \ge 0$.

2. A counterexample. To verify our statement in the last lines of §1, we argue for the case $d = 1$ and $\Omega = (0, 1)$, the unit open interval.

Proposition 2.1. *Let $d = 1$ and $\Omega = (0,1)$. There is a bounded sequence $\{u_n(x)\} \subset \mathcal{H}^1(0,1)$*

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which converges to an element $u(x)$ effectively in $\mathcal{H}^0(0,1)$ but non-effectively in $\mathcal{H}^1(0,1)$.

Note that the limit $u(x) \in \mathcal{H}^0(0,1)$ actually belongs to $\mathcal{H}^1(0,1)$ because of weak compactness. In fact, we can then extract a subsequence of $\{u_n(x)\}$ which converges weakly to some element $\tilde{u}(x)$ in $\mathcal{H}^1(0,1)$. By Rellich's theorem, this subsequence converges to $\tilde{u}(x)$ in $\mathcal{H}^0(0,1)$. However, the subsequence already converges to $u(x)$ in $\mathcal{H}^0(0,1)$, and thus $\tilde{u}(x) = u(x)$.

To achieve the proof of the proposition, we adopt the idea of Pour-El and Richards [5], Chapter 1 §1 (p.52). Let

$$\varphi(t) = \begin{cases} \exp\left(-\dfrac{t^2}{1-t^2}\right), & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}.$$

$\varphi(t)$ is a non-negative $\mathcal{C}^\infty$ even function and its support is the closed interval $[-1,1]$.

Let $a : \mathbf{N} \to \mathbf{N}$ be a one-to-one recursive function which enumerates a recursively enumerable non-recursive set A. We may assume $0 \notin A$ or $a(n) > 0$ for all n. Now put

$$\varphi_n(x) = \varphi\big(2^{(n+a(n)+2)}\,(x - 2^{-a(n)})\big). \tag{3}$$

Each $\varphi_n(x)$ is supported on a closed subinterval

$$\big[2^{-a(n)} - 2^{-(n+a(n)+2)},\, 2^{-a(n)} + 2^{-(n+a(n)+2)}\big]$$

of $(0,1)$. $\varphi_n(x)$ and $\varphi_{n'}(x)$ have disjoint supports for $n \neq n'$. For, we may assume without loss of generality that $a(n) < a(n') = a(n) + k$ for some $k \geq 1$. Then disjointness of the supports of $\varphi_n(x)$ and $\varphi_{n'}(x)$ reduces to positivity of the difference

$$\begin{aligned} &\big(2^{-a(n)} - 2^{-(n+a(n)+2)}\big) \\ &\quad - \big(2^{-a(n')} + 2^{-(n'+a(n')+2)}\big) \\ &= 2^{-(a(n')+2)}\big(2^{k+2} - 2^{k-n} - 2^2 - 2^{-n'}\big). \end{aligned}$$

However,

$$2^{k+2} - 2^{k-n} - 2^2 - 2^{-n'} \geq 3 \cdot 2^k - 5 > 0$$

since $n,\, n' \geq 0$ and $k \geq 1$.

The $\mathcal{L}^2$-norms of $\varphi_n(x)$ and its derivative $\varphi_n'(x)$ are given by

$$\|\varphi_n\|_0^2 = 2^{-(n+a(n)+2)}\, c_0, \tag{4}$$

$$\|\varphi_n'\|_0^2 = 2^{n+a(n)+2}\, c_1, \tag{5}$$

where

$$c_0 = 2\int_0^1 \varphi(t)^2\,dt, \quad c_1 = 2\int_0^1 \varphi'(t)^2\,dt$$

are both computable reals.

Let

$$u_n(x) = \sum_{k=0}^{n} 2^{-b(k)}\,\varphi_k(x), \quad n = 0,1,2,\cdots. \tag{6}$$

Choosing $b(k)$ appropriately, we will have the proposition verified. Let us compute the $\mathcal{H}^m(0,1)$-norms of $u_n(x)$ for $m = 0,1$. The orthogonality then implies

$$\begin{aligned} \|u_n\|_0^2 &= \sum_{k=0}^{n} 2^{-2b(k)}\,\|\varphi_k\|_0^2 \\ &= c_0 \sum_{k=0}^{n} 2^{-2b(k)-k-a(k)-2}. \end{aligned} \tag{7}$$

In particular, for whatever $a(k) > 0$ and $b(k) > 0$, the sequence $\{u_n(x)\}$ converges effectively to the element

$$u(x) = \sum_{k=0}^{\infty} 2^{-b(k)}\,\varphi_k(x) \tag{8}$$

in $\mathcal{L}^2(0,1)$. In fact, we have

$$\begin{aligned} \|u - u_n\|_0^2 &= c_0 \sum_{k=n+1}^{\infty} 2^{-2b(k)-k-a(k)-2} \\ &< 2^{-n-1}\,c_0, \end{aligned}$$

since $a(k) + 2b(k) > 0$. On the other hand, note

$$\begin{aligned} \|u_n'\|_0^2 &= \sum_{k=0}^{n} 2^{-2b(k)}\,\|\varphi_k'\|_0^2 \\ &= c_1 \sum_{k=0}^{n} 2^{-2b(k)+k+a(k)+2}. \end{aligned} \tag{9}$$

Therefore, taking

$$b(k) = a(k) + \frac{1}{2}k,$$

we see that $\{u_n'(x)\}$ converges to

$$v(x) = \sum_{k=0}^{\infty} 2^{-b(k)}\,\varphi_k'(x) \tag{10}$$

in $\mathcal{L}^2(0,1)$ since

$$\|v - u_n'\|_0^2 = c_1 \sum_{k=n+1}^{\infty} 2^{-a(k)+2}.$$

However, this convergence is not effective (See [5], p. 16). It is readily seen that $v(x)$ is the weak derivative $u'(x)$ of $u(x)$, whence $u \in \mathcal{H}^1(0,1)$. Then the sequence $\{u_n(x)\}$ converges to $u(x)$ in $\mathcal{H}^1(0,1)$ as

$$\|u - u_n\|_1^2 = \|u - u_n\|_0^2 + \|v - u_n'\|_0^2 \;\to\; 0, \quad n \to \infty.$$

This convergence is not effective.

The weak convergence of the sequence $\{u_n(x)\}$ is not effective in the following sense.

Corollary 2.1. *There is a* $\hat{u}(x) \in \mathcal{H}^1(0,1)$ *such that* $(u_n - u\,,\, \hat{u})_1$ *does not converge effectively.*

In fact, take $\hat{u}(x) = u(x)$. Then

$$(u - u_n,\, u)_1 = (u,\, u)_1 - (u_n,\, u_n\,)_1 = \|u - u_n\|_1^2$$

because of disjointness of the supports of $\varphi_k(x)$.

Remark 2.1. Analogously to (5), $\mathcal{L}^2$-norms of the m-th derivatives $\varphi_n^{(m)}(x)$ of $\varphi_n(x)$ $(m = 2, 3, \cdots)$ are given by

$$\|\varphi_n^{(m)}\|_0^2 = 2^{(2m-1)(n+a(n)+2)}\, c_m,$$
$$c_m = 2\int_0^1 \varphi^{(m)}(t)^2\, dt,$$

where c_m are computable. Therefore, taking

$$b(k) = m\, a(k) + \left(m - \frac{1}{2}\right) k$$

in (6), we have a non-effectively convergent sequence $\{u_n(x)\}$ in $\mathcal{H}^m(0,1)$ which converges effectively in $\mathcal{H}^0(0,1)$. $\{u_n(x)\}$ also converges effectively in each of $\mathcal{H}^l(0,1)$, $\; m > l \geq 0$.

3. Further observation. Let $0 < s < 1$. The Sobolev space $\mathcal{H}^s(0,1)$ of order s can be defined via the Fourier series expansions. Let $w(x) \in \mathcal{L}^2(0,1)$ be expanded into the Fourier series

$$w(x) = \alpha_0 + \sum_{n=1}^{\infty} \{\alpha_n \cos 2n\pi x + \beta_n \sin 2n\pi x\}.$$

Then we have $w \in \mathcal{H}^s(0,1)$ if and only if

$$(11) \quad |\alpha_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(1+n^2)^s\{|\alpha_n|^2 + |\beta_n|^2\} < +\infty.$$

In fact, (11) gives the square $\|w\|_s^2$ of the $\mathcal{H}^s(0,1)$-norm of $w(x)$.

Observe that we have the logarithmic convexity of norms

$$(12) \qquad \|w\|_s \leq \|w\|_0^{1-s}\, \|w\|_1^s \quad (0 < s < 1)$$

for $w \in \mathcal{H}^1(0,1)(\subset \mathcal{H}^s(0,1) \subset \mathcal{H}^0(0,1))$. In fact, it is easy to see (12) in the present case. For we have

$$(1+n^2)^s \leq (1-s)\,\epsilon^{-s} + s\,\epsilon^{1-s}\,(1+n^2)$$

for all $\epsilon > 0$ and $n = 0, 1, 2, \cdots$. Thus, (11) implies that if $w \in \mathcal{H}^1(0,1)$, then

$$\|w\|_s^2 \leq (1-s)\,\epsilon^{-s}\,\|w\|_0^2 + s\,\epsilon^{1-s}\,\|w\|_1^2$$

for all $\epsilon > 0$. Taking the minimum of the right hand side, we get (12).

The space $\mathcal{H}^s(0,1)$ is obtained as the complex interpolation space $\mathcal{H}^s(0,1) = [\mathcal{H}^0(0,1)\,, \mathcal{H}^1(0,1)]_s$ in the sense of Calderón [3]. (See, e.g., Bergh *et al*. [2]). Then recall that the computability structure in $\mathcal{H}^s(0,1)$ is induced from those of $\mathcal{H}^0(0,1)$ and $\mathcal{H}^1(0,1)$ if s is computable (See Yoshikawa [6]).

Proposition 3.1. *Let* $0 < s < 1$ *be computable. Then the sequence* $\{u_n(x)\} \subset \mathcal{H}^1(0,1)$ *in Proposition* 2.1 *effectively converges to* $u(x)$ *also in* $\mathcal{H}^s(0,1)$.

In fact, from (12), we have

$$\|u - u_n\|_s \leq \|u - u_n\|_0^{1-s}\, \|u - u_n\|_1^s.$$

Note

$$\|u - u_n\|_1 < \sqrt{c_0 + 4c_1} = c.$$

Hence,

$$\|u - u_n\|_s \leq c^s\, \|u - u_n\|_0^{1-s} = 2^{-(1-s)(n+1)}\, c^s.$$

Thus, choose a recursive function $e_s(N)$ such that

$$e_s(N) \geq \frac{N}{1-s} + \frac{s}{1-s}\log_2 c - 1.$$

Then we have $\|u - u_n\|_s < 2^{-N}$ for $n > e_s(N)$.

Remark 3.1. We may take $e_s(N) \geq e_{s'}(N)$, $s > s'$, since $\|u - u_n\|_{s'} \leq \|u - u_n\|_s$ if $s > s' \geq 0$.

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