Poincaré formulas of complex submanifolds

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(Communicated by Heisuke HIRONAKA, M. J. A., June 15, 2004)

Abstract: We formulate Poincaré formulas of complex submanifolds in almost Hermitian homogeneous spaces, using Howard's formulation of Poincaré formulas in Riemannian homogeneous spaces. This formula is an extension of Santaló's one in complex space forms.

Key words: Poincaré formula; complex submanifolds; almost Hermitian homogeneous spaces.

1. Introduction. Let M and N be submanifolds in a Riemannian homogeneous space G/K. Then many works in integral geometry have been concerned with computing integrals of the following form

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g).$$

The Poincaré formula means equalities which represent the above integral by some geometric invariants of submanifolds M and N of G/K.

Santaló [3] showed that if M and N are complex submanifolds in a complex space form G/K which satisfy $\dim M + \dim N \geq \dim(G/K)$ then the Poincaré formula is expressed as a constant times the product of the volumes of M and N. Howard [1] obtained the generalized Poincaré formula in Riemannian homogeneous spaces, and he reformulated Santaló's Poincaré formula.

In the present paper, we attempt to describe this formula for complex submanifolds M and N of almost Hermitian homogeneous spaces G/K. And we show that the Poincaré formula can be expressed as a constant times the product of the volumes of M and N if K acts irreducibly on an exterior algebra (Theorem 3.1).

The second named author [2] recently extend the main result of the present article in the case of irreducible Hermitian symmetric spaces.

2. Preliminaries. In this section, we shall review the generalized Poincaré formula in Riemannian homogeneous spaces obtained by Howard [1].

Let E be a finite dimensional real vector space with an inner product. For two vector subspaces V and W of dimension p and q in E, take orthonormal bases $v_1, \dots, v_p$ and $w_1, \dots, w_q$ of V and W respectively, and define

$$\sigma(V, W) = \|v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q\|.$$

This definition is independent of the choice of orthonormal bases. Furthermore, if $p + q = \dim E$, then

$$\sigma(V, W) = \sigma(V^\perp, W^\perp).$$

where $V^\perp$ and $W^\perp$ are the orthogonal complements of V and W respectively.

Let G be a unimodular Lie group and K a closed subgroup of G. We assume that G has a left invariant Riemannian metric that is also invariant under the right actions of elements of K. This metric induces a G-invariant Riemannian metric on G/K. We denote by o the origin of G/K. For x and y in G/K and vector subspaces V and W in $T_x(G/K)$ and $T_y(G/K)$, we define $\sigma_K(V, W)$ by

(2.1)
$$\sigma_K(V, W) = \int_K \sigma((dg_x)_o^{-1} V, dk_o^{-1} (dg_y)_o^{-1} W) d\mu_K(k)$$

where g_x and g_y are elements of G such that $g_x o = x$ and $g_y o = y$. This definition is independent of the choice of g_x and g_y in G such that $g_x o = x$ and $g_y o = y$. With these facts, the generalized Poincaré formula in homogeneous spaces can be stated,

2000 Mathematics Subject Classification. 53C65.

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(2.2)
$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \int_{M\times N} \sigma_K(T_x^\perp M, T_y^\perp N) d\mu_{M\times N}(x,y).$$

3. Poincaré formulas of complex submanifolds.

Theorem 3.1. *Let G be a unimodular Lie group and G/K an almost Hermitian homogeneous space of complex dimension n. Assume that K acts irreducibly on $\wedge^p(T_o(G/K))^{(1,0)}$. Take two almost complex submanifolds M and N of G/K with*

$$\dim_{\mathbf{C}} M = p, \quad \dim_{\mathbf{C}} N = n-p.$$

Then we have

$$\int_G \sharp(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(K)}{\binom{n}{p}} \mathrm{vol}(M)\mathrm{vol}(N).$$

Proof. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively, let $\mathfrak{m}$ denote some vector subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ (direct sum decomposition). Then we can identify $\mathfrak{m}$ with $T_o(G/K)$ in the natural manner. We denote by J and $\langle \cdot, \cdot \rangle$ the complex structure and the inner product on $\mathfrak{m}$ induced from the almost complex structure and from the G-invariant Riemannian metric on G/K respectively.

Let $\{u_i, Ju_i\}_{i=1}^p$ and $\{v_i, Jv_i\}_{i=1}^p$ be orthonormal bases of $(dg_x)^{-1}(T_x(M))$ and $(dg_y)^{-1}(T_y^\perp(N))$ respectively. We define

$$\xi_i = \frac{1}{\sqrt{2}}(u_i - \sqrt{-1}Ju_i), \quad \eta_i = \frac{1}{\sqrt{2}}(v_i - \sqrt{-1}Jv_i),$$

then $\{\xi_i\}_{i=1}^p$ is a unitary basis of the tangent space $(dg_x)^{-1}(T_x(M))^{(1,0)}$ of type $(1,0)$ and $\{\eta_i\}_{i=1}^p$ is that of $(dg_y)^{-1}(T_y^\perp(N))^{(1,0)}$. Hence we have the following equation:

$$\begin{aligned} &u_1 \wedge \cdots \wedge u_p \wedge Ju_1 \wedge \cdots \wedge Ju_p \\ &\quad = (\sqrt{-1})^p \xi_1 \wedge \cdots \wedge \xi_p \wedge \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_p, \\ &v_1 \wedge \cdots \wedge v_p \wedge Jv_1 \wedge \cdots \wedge Jv_p \\ &\quad = (\sqrt{-1})^p \eta_1 \wedge \cdots \wedge \eta_p \wedge \bar{\eta}_1 \wedge \cdots \wedge \bar{\eta}_p. \end{aligned}$$

We extend $\langle \cdot, \cdot \rangle$ to a complex bilinear form on $\mathfrak{m}^{\mathbf{C}}$, and denote by the same symbol. We note that if X and Y are both in the $\mathfrak{m}^{(1,0)}$ (or $\mathfrak{m}^{(0,1)}$) then $\langle X, Y\rangle = 0$. So we have

$$\begin{aligned} &\big\langle u_1 \wedge \cdots \wedge u_p \wedge Ju_1 \wedge \cdots \wedge Ju_p, \\ &\ \mathrm{Ad}(k)(v_1 \wedge \cdots \wedge v_p \wedge Jv_1 \wedge \cdots \wedge Jv_p)\big\rangle \\ &\quad = \big\langle \xi_1 \wedge \cdots \wedge \xi_p \wedge \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_p, \\ &\qquad \overline{\mathrm{Ad}(k)(\eta_1 \wedge \cdots \wedge \eta_p \wedge \bar{\eta}_1 \wedge \cdots \wedge \bar{\eta}_p)}\big\rangle \\ &\quad = \det \begin{bmatrix} \langle \xi_i, \overline{\mathrm{Ad}(k)\eta_j}\rangle & \langle \xi_i, \mathrm{Ad}(k)\eta_j\rangle \\ \langle \bar{\xi}_i, \overline{\mathrm{Ad}(k)\eta_j}\rangle & \langle \bar{\xi}_i, \mathrm{Ad}(k)\eta_j\rangle \end{bmatrix} \\ &\quad = \left|\det\left[\langle \xi_i, \overline{\mathrm{Ad}(k)\eta_j}\rangle\right]\right|^2 \\ &\quad \geq 0. \end{aligned}$$

From (2.1) we have

$$\begin{aligned} &\sigma_K(T_x^\perp M, T_y^\perp N) \\ &\quad = \int_K \left|\det\left[\langle \xi_i, \overline{\mathrm{Ad}(k)\eta_j}\rangle\right]\right|^2 d\mu_K(k) \\ &\quad = \int_K \left|\langle \xi_1 \wedge \cdots \wedge \xi_p, \overline{\mathrm{Ad}(k)(\eta_1 \wedge \cdots \wedge \eta_p)}\rangle\right|^2 \\ &\qquad \cdot d\mu_K(k). \end{aligned}$$

Fix $\eta = \eta_1 \wedge \cdots \wedge \eta_p$, and for any X and Y in $\wedge^p(\mathfrak{m}^{(1,0)})$ we define

$$Q_\eta(X,Y) = \int_K \langle X \wedge \bar{Y}, \overline{\mathrm{Ad}(k)(\eta \wedge \bar{\eta})}\rangle d\mu_K(k).$$

Then Q_η is a Hermitian form on $\wedge^p(\mathfrak{m}^{(1,0)})$ invariant by K-action. From Schur's lemma, since K acts irreducibly on $\wedge^p(\mathfrak{m}^{(1,0)})$, there exists a positive constant C_η such that

$$Q_\eta(X,Y) = C_\eta \langle X, Y\rangle.$$

Hence we get

$$\sigma_K(T_x^\perp M, T_y^\perp N) = Q_\eta(\xi,\xi) = C_\eta.$$

This implies $\sigma_K(T_x^\perp M, T_y^\perp N)$ is independent of $T_x^\perp M$. Similarly $\sigma_K(T_x^\perp M, T_y^\perp N)$ is also independent of $T_y^\perp N$. Thus $\sigma_K(T_x^\perp M, T_y^\perp N)$ is constant C.

Let $\{X_i\}_{i=1}^r$ be a unitary basis of $\wedge^p(\mathfrak{m}^{(1,0)})$, and put

$$r = \dim(\wedge^p(\mathfrak{m}^{(1,0)})) = \binom{n}{p}.$$

Since for any i and j

$$\int_K \left|\langle X_i, \overline{\mathrm{Ad}(k)X_j}\rangle\right|^2 d\mu_K(k) = C,$$

we have

$$\begin{aligned} rC &= \sum_{i=1}^r \int_K \left|\langle X_i, \overline{\mathrm{Ad}(k)X_1}\rangle\right|^2 d\mu_K(k) \\ &= \int_K \sum_{i=1}^r \left|\langle X_i, \overline{\mathrm{Ad}(k)X_1}\rangle\right|^2 d\mu_K(k) \\ &= \int_K \|\mathrm{Ad}(k)X_1\|^2 d\mu_K(k) \\ &= \mathrm{vol}(K). \end{aligned}$$

This completes the proof. □

Corollary 3.2. *Let G/K be an irreducible Hermitian symmetric space. Let M be a complex curve and N a complex hypersurface of G/K. Then we have*

$$\int_G \sharp(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(K)}{\dim_{\mathbf{C}}(G/K)} \mathrm{vol}(M)\mathrm{vol}(N).$$

This corollary in the case where N is the cut locus of a point in G/K has been already obtained in [4], although the constant is expressed in a different way.

If $p > 1$, then we can have many examples which holds the situation of Theorem 3.1. In the case where G/K is an irreducible Hermitian symmetric space, we can give $p\,(> 1)$ when K acts irreducibly on

$$\wedge^p (T_o(G/K))^{(1,0)}$$

in the following table:

Compact type	
$A\ III$	$SU(l)/S(U(m) \times U(l-m))$
	(any p (if $m = 1$), empty (if $m \neq 1$))
$D\ III$	$SO(2l)/U(l)$ $(p = 2)$
$BD\ I$	$SO(2l)/SO(2) \times SO(2l-2)$
	$(p \neq l-1)$
	$SO(2l+1)/SO(2) \times SO(2l-1)$
	(any p)
$C\ I$	$Sp(l)/U(l)$ $(p = 2)$
$E\ III$	$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + \mathbf{R})$ $(p = 2, 3)$
$E\ VII$	$(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6 + \mathbf{R})$ $(p = 2, 3, 4)$

For the rest of this section we will consider the case where $\wedge^p (T_o(G/K))^{(1,0)}$ is reducible by K-action. The other conditions are same with Theorem 3.1.

$$\wedge^p (T_o(G/K))^{(1,0)} = \bigoplus_{i=1}^{s} V_i$$

denotes the irreducible decomposition by K-action. Let X and Y be complex vector subspaces of dimension p in $(T_o(G/K))^{(1,0)}$ and take unitary bases $\{\xi_i\}_{i=1}^p$ and $\{\eta_i\}_{i=1}^p$ of X and Y respectively. We denote by $\hat{X}_i$ and $\hat{Y}_i$ be the V_i-components of $\xi_1 \wedge \cdots \wedge \xi_p$ and $\eta_1 \wedge \cdots \wedge \eta_p$ respectively. We define $A(X, Y)$ by

$$A(X, Y) = \sum_{i=1}^{s} \frac{\mathrm{vol}(K)}{\dim V_i} \|\hat{X}_i\|^2 \|\hat{Y}_i\|^2.$$

Theorem 3.3. *If V_i and V_j are not equivalent when $i \neq j$, then we have*

$$\int_G \sharp(M \cap gN) d\mu_G(g) = \int_{M\times N} A(T_x M, T_y^{\perp} N) d\mu_{M\times N}(x, y).$$

Proof. From the proof of Theorem 3.1, we have

$$\begin{aligned}
&\sigma_K(X, Y) \\
&= \int_K \left| \langle \xi_1 \wedge \cdots \wedge \xi_p,\ \overline{\mathrm{Ad}(k)(\eta_1 \wedge \cdots \wedge \eta_p)} \,\rangle \right|^2 \\
&\quad \cdot d\mu_K(k) \\
&= \int_K \left| \sum_{i=1}^{s} \langle \hat{X}_i,\ \overline{\mathrm{Ad}(k)\hat{Y}_i} \,\rangle \right|^2 d\mu_K(k) \\
&= \sum_{i,j=1}^{s} \int_K \langle \hat{X}_i,\ \overline{\mathrm{Ad}(k)\hat{Y}_i} \,\rangle \overline{\langle \hat{X}_j,\ \overline{\mathrm{Ad}(k)\hat{Y}_j} \,\rangle} \\
&\quad \cdot d\mu_K(k).
\end{aligned}$$

The last integrals vanish when $i \neq j$ by the Peter-Weyl theorem. Therefore we get

$$\sigma_K(X, Y) = \sum_{i=1}^{s} \int_K \left| \langle \hat{X}_i,\ \overline{\mathrm{Ad}(k)\hat{Y}_i} \,\rangle \right|^2 d\mu_K(k).$$

By the similar way with Theorem 3.1, we can conclude that each integral in just above equation is constant and determine it.

$$\sigma_K(X, Y) = \sum_{i=1}^{s} \frac{\mathrm{vol}(K)}{\dim V_i} \|\hat{X}_i\|^2 \|\hat{Y}_i\|^2 = A(X, Y).$$

□

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