On the Diophantine equation $x(x+1)\cdots(x+n)+1=y^2$ $(17 \le n = \text{odd} \le 27)$

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Abstract: We consider the Diophantine equation as mentioned in the title and solve it completely, i.e., show that there exist no integer solution satisfying this equation.

Key word: Diophantine equation.

1. Introduction. Erdös and Selfridge [2] proved that the Diophantine equation $x(x+1)\cdots(x+n)=y^2$ has no positive integer solution. Abe [1] considered the following modified equation. Let $\mathbf{N}$ be the set of all positive integers. Abe found all $(x,y)\in\mathbf{N}^2$ satisfying the Diophantine equation $x(x+1)\cdots(x+n)+1=y^2$ for odd integer n such that $1\le n\le 15$. His results are as follows: For $n=3$, $x(x+1)(x+2)(x+3)+1=(x^2+3x+1)^2$. So for any $x\in\mathbf{N}$, (x, x^2+3x+1) are solutions. For $n=5$, there is only one solution $(2, 71)$. For $n=1$ or $7\le n\le 15$, there exist no solution. In this paper we shall extend this for the case $17\le n\le 27$ and prove that there exist no positive integer solution using computer.

2. A principle and results. Let n be an odd positive integer and $F(x)$ be

$$F(x)=x(x+1)(x+2)\cdots(x+n)+1.$$

Then $F(x)$ is a monic integral polynomial of an even degree $2m$, where $m=(n+1)/2$. We can obtain a monic polynomial

$$G(x)=x^m+a_1x^{m-1}+\cdots+a_m\in\mathbf{Q}[x]$$

and another polynomial $R(x)\in\mathbf{Q}[x]$ whose degree $\deg R(x)<m$, such that

$$F(x)=G(x)^2+R(x).$$

In fact the denominator of the coefficient of $G(x)$ is a power of 2. We shall denote by ε the inverse number of the maximum of these denominators. Using computer we get next result

n	$G(x)$	ε
17	$\{2H(x)+1\}/2^{16}$	$1/2^{16}$
19	$\{2H(x)+x(x+1)+1\}/2^{3}$	$1/2^{3}$
21	$\{2H(x)+1\}/2^{19}$	$1/2^{19}$
23	$H(x)$	1
25	$\{2H(x)+1\}/2^{23}$	$1/2^{23}$
27	$\{2H(x)+x(x+1)+1\}/2^{4}$	$1/2^{4}$

for some $H(x)\in\mathbf{Z}[x]$. When $\varepsilon<1$ we have $G(x)=$ (odd number) $\cdot\,\varepsilon$ for any integer x. We shall put $G_r(x)$ and $Y_r(x)$ as

$$G_r(x)=G(x)-(2r-1)\varepsilon, \quad \text{when} \quad \varepsilon<1$$
$$G_r(x)=G(x)-r, \quad \text{when} \quad \varepsilon=1$$
$$Y_r(x)=[G_r(x)]$$

for integer r such that $0\le r\le \max r$ where $\max r$ are

n	$\max r$
17	76560
19	1
21	2262103
23	1
25	194885048
27	1289

In this range all coefficients of $G_r(x)$ are positive. When $\varepsilon<1$ we have $G_r(x)=$ (even number) $\cdot\,\varepsilon$ for any integer x. Therefore for any positive integer x we have

(1)

$$Y_r(x)\le G_r(x)<G_{r-1}(x)=G_r(x)+2\varepsilon\le Y_r(x)+1$$
$$\text{when} \quad \varepsilon<1.$$

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(2)

$$Y_r(x) = G_r(x) < G_{r-1}(x) = G_r(x) + 1 = Y_r(x) + 1 \quad \text{when} \quad \varepsilon = 1.$$

Using computer we have all coefficients of $F(x) - G_0(x)^2$ are negative and for $1 \leq r \leq \max r$

$$F(x) - G_r(x)^2 = b_0 x^m - b_1 x^{m-1} - \cdots - b_m \quad \text{when} \quad n = 17,\ 21,\ 25.$$

$$F(x) - G_r(x)^2 = b_0 x^m + b_1 x^{m-1} - \cdots - b_m \quad \text{when} \quad n = 19,\ 23,\ 27.$$

for some positive rational numbers b_i. Therefore there exists only one positive real root α_r for the equation $F(x) - G_r(x)^2 = 0$ by Descartes' rule. Using Newton's method we find that all α_r are not integers. We shall put $x_r = [\alpha_r]$. Then we have for positive integer x

(3) $$x_1 < x \Rightarrow G_1(x)^2 < F(x) < G_0(x)^2.$$

(4) $$x_r < x \leq x_{r-1} \Rightarrow G_r(x)^2 < F(x) < G_{r-1}(x)^2.$$

From (1)∼(4) we get for positive integer x

$$x_1 < x \Rightarrow Y_1(x)^2 < F(x) < (Y_1(x) + 1)^2.$$

$$x_r < x \leq x_{r-1} \Rightarrow Y_r(x)^2 < F(x) < (Y_r(x) + 1)^2.$$

Therefore we have no positive integer solution of $F(x) = y^2$ for $x > x_{\max r}$. Using computer we have

n	x_1	$x_{\max r}$
17	153119304151	999993
19	56145	56145
21	452420485347120	99999986
23	464066	464066
25	3897700942901197318	9999999969
27	50749688	999701
29	23060745354661304625864	

When $x \leq x_{\max r}$, we can prove that $F(x) = y^2$ has no positive integer solution using computer.

When $n = 25$, we used a personal computer about two weeks for getting the result. For $n = 29$, we found that x_1 is too large. So we could not continue.

References

[1] Abe, N.: On the Diophantine equation $x(x+1)\cdots(x+n)+1 = y^2$. Proc. Japan Acad., **76A**, 16–17 (2000).

[2] Erdös, P., and Selfridge, J. L.: The product of consecutive integers is never a power. Illinois J. Math., **19**, 292–301 (1975).