Flips in dimension three via crepant descent method

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Abstract: We prove the existence of 3-dimensional flips by using the crepant descent method. Our proof depends on the existence of good members in the anticanonical linear system and uses explicit computations of blowing ups of terminal singularities.

Key words: Flip; flop; terminal singularity.

1. Introduction. In this article, we shall give a proof of the existence of flips in dimension three. This was first proved by [15] and completed the Minimal Model Program. The method in [15] was to find a good member in $|-K|$ or $|-2K|$ and construct a double cover by using this. Here we shall give another approach, which is based on the existence of a good member in $|-K|$ and some computations of blowing ups of terminal singularities. The material here is sketched in [2, 1.6].

We shall outline our proof briefly. Let $f : X \to Z$ be a flipping contraction. We want to construct a flip $f^+ : X^+ \to Z$ of f. Let $\mathcal{H}_X = |-K_X|$, then there is a member $S \in \mathcal{H}_X$ such that S has only rational double points. This is proved by [12, 2.1] assuming that the exceptional locus of f is an irreducible curve, and by Mori (unpublished) in general. Hence the pair $(X, \mathcal{H}_X)$ is canonical, and the existence of flips is a consequence of the existence of flops with respect to $K_X + \mathcal{H}_X$. Our aim here is to construct canonical flops with respect to $K_X + \mathcal{H}_X$. In order to construct such flops, we use the crepant descent method explained in [11, Chap. 6]. The proof is done by induction on $e(X, \mathcal{H}_X)$, the number of crepant prime divisors with respect to $K_X + \mathcal{H}_X$. By explicit computations of blowing ups of terminal singularities, we construct a pair $(Y, \mathcal{H}_Y)$ and a birational morphism $q : Y \to X$ such that $e(Y, \mathcal{H}_Y) = e(X, \mathcal{H}_X) - 1$. We can reduce the problem on $(X, \mathcal{H}_X)$ to the problem on $(Y, \mathcal{H}_Y)$. The argument here is the same as in [11, Chap. 6]. If $e(X, \mathcal{H}_X) = 0$, then the pair $(X, \mathcal{H}_X)$ is terminal, and in particular X has only Gorenstein terminal singularities. In this case, flops with respect to $K_X + \mathcal{H}_X$ are usual terminal flops. See (2.5) and (2.6) for details.

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This article is organized as follows: In Section 2, we discuss about the existence and the termination of flops for a canonical pair $(X, \mathcal{H}_X)$. Necessary definitions are included here. Section 3 deals with the existence of flips as an application of the results in the previous section. In Section 4, we discuss about blowing ups of 3-dimensional terminal singularities, which are the basis for the crepant descent method in Section 2. We always work over the complex number field $\mathbf{C}$.

2. Flops for canonical pairs. Let X be a normal $\mathbf{Q}$-factorial 3-fold and let $\mathcal{H}_X$ be a movable linear system on X without fixed components.

2.1. Let $\mu : \tilde{X} \to X$ be a resolution of singularities of X such that the birational transform $\mathcal{H}_{\tilde{X}} = \mu_*^{-1}\mathcal{H}_X$ is free. We write

$$K_{\tilde{X}} + \mathcal{H}_{\tilde{X}} = \mu^*(K_X + \mathcal{H}_X) + \sum a(E, X, \mathcal{H}_X)E,$$

where the sum runs over all exceptional prime divisors E of μ and $a(E, X, \mathcal{H}_X) \in \mathbf{Q}$. We call $a(E, X, \mathcal{H}_X)$ the *discrepancy* of E over X with respect to $K_X + \mathcal{H}_X$. This depends only on the discrete valuation on the function field $\mathbf{C}(X)$ of X determined by E, and does not depend on the particular choice of the resolution μ. Thus $a(E, X, \mathcal{H}_X)$ is defined for all exceptional prime divisors E over X.

We say that $K_X + \mathcal{H}_X$ (or the pair $(X, \mathcal{H}_X)$) is *canonical* (resp. *terminal*) if $a(E, X, \mathcal{H}_X) \geq 0$ (resp. $a(E, X, \mathcal{H}_X) > 0$) for all exceptional prime divisors E over X. In this case, we also say that $(X, \mathcal{H}_X)$ is a *canonical* (resp. *terminal*) *pair*.

For a canonical pair $(X, \mathcal{H}_X)$, we define

$$e(X, \mathcal{H}_X) = \#\left\{E : \begin{array}{c}\text{exceptional prime}\\ \text{divisor over } X\end{array} \;\middle|\; a(E, X, \mathcal{H}_X) = 0\right\}$$

and call it the number of *crepant divisors* over X with respect to $K_X + \mathcal{H}_X$. We easily see that $e(X, \mathcal{H}_X) < +\infty$ and that $(X, \mathcal{H}_X)$ is terminal if and only if $e(X, \mathcal{H}_X) = 0$. It is also obvious that if $(X, \mathcal{H}_X)$ is canonical (resp. terminal), then X itself has only canonical (resp. terminal) singularities.

2.2. Let $(X, \mathcal{H}_X)$ be a canonical pair. Let $f : X \to Z$ be a projective birational morphism onto a normal 3-fold Z and let D be an effective $\mathbf{Q}$-divisor on X.

We say that $f : X \to Z$ is a *D-flopping contraction* with respect to $K_X + \mathcal{H}_X$ if $\rho(X/Z) = 1$, $\dim \operatorname{Exc}(f) = 1$, $K_X + \mathcal{H}_X$ is f-trivial and $-D$ is f-ample, where $\operatorname{Exc}(f)$ denotes the exceptional locus of f.

Let $f : X \to Z$ be a D-flopping contraction. A projective birational morphism $f^+ : X^+ \to Z$ from a normal 3-fold X^+ is called a *D-flop* of f with respect to $K_X + \mathcal{H}_X$ if $\rho(X^+/Z) = 1$, $\dim \operatorname{Exc}(f^+) = 1$, $K_{X^+} + \mathcal{H}_{X^+}$ is f^+-trivial and D^+ is f-ample, where D^+ (resp. $\mathcal{H}_{X^+}$) is the birational transform of D (resp. $\mathcal{H}_X$). It is easy to see that such a morphism $f^+ : X^+ \to Z$ is unique if it exists. The birational map $X \dashrightarrow X^+$ is also called a D-flop with respect to $K_X + \mathcal{H}_X$.

We shall omit the phrase "with respect to $K_X + \mathcal{H}_X$" if there is no danger of confusion.

2.3. Let $(X, \mathcal{H}_X)$ be a canonical pair and let D be an effective $\mathbf{Q}$-divisor on X.

We say that the D-flop *exists* if the D-flop $f^+ : X^+ \to Z$ exists for any D-flopping contraction $f : X \to Z$, and that a sequence of D-flops for $(X, \mathcal{H}_X)$ *terminates* if there exists no infinite sequence

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots ,$$

where $X_i \dashrightarrow X_{i+1}$ is a D_i-flop with respect to $K_{X_i} + \mathcal{H}_{X_i}$, and D_i (resp. $\mathcal{H}_{X_i}$) is the birational transform of D (resp. $\mathcal{H}_X$).

Lemma 2.4. *Let $(X, \mathcal{H}_X)$ be a canonical pair and let D be an effective $\mathbf{Q}$-divisor on X. Let $X \dashrightarrow X^+$ be the D-flop with respect to $K_X + \mathcal{H}_X$. Then $a(E, X, \mathcal{H}_X) = a(E, X^+, \mathcal{H}_{X^+})$ for each exceptional prime divisor E over X, where $\mathcal{H}_{X^+}$ is the birational transform of $\mathcal{H}_X$. In particular, $(X^+, \mathcal{H}_{X^+})$ is canonical and $e(X, \mathcal{H}_X) = e(X^+, \mathcal{H}_{X^+})$.*

Proof. This follows from [11, 2.28]. □

Lemma 2.5. *Let $(X, \mathcal{H}_X)$ be a canonical pair. Then we have the following*:

(1) *$(X, \mathcal{H}_X)$ is terminal if and only if X has only terminal singularities and the base locus $\operatorname{Bs} \mathcal{H}_X$ of $\mathcal{H}_X$ consists only of isolated nonsingular points P of X such that $\operatorname{mult}_P \mathcal{H}_X = 1$.*

(2) *If $(X, \mathcal{H}_X)$ is terminal and if $\mathcal{H}_X \subseteq |-K_X|$, then X has only Gorenstein terminal singularities.*

Proof. (1) follows from [1, 1.22]. If $\mathcal{H}_X \subseteq |-K_X|$, then every point $P \in X$ with index ≥ 2 is contained in $\operatorname{Bs} \mathcal{H}_X$, and hence (2) follows from (1). □

Proposition 2.6. *Let $(X, \mathcal{H}_X)$ be a terminal pair with $\mathcal{H}_X \subseteq |-K_X|$, and let D be an effective $\mathbf{Q}$-divisor on X. Then the D-flop of $(X, \mathcal{H}_X)$ exists and a sequence of D-flops for $(X, \mathcal{H}_X)$ terminates.*

Proof. Let $f : X \to Z$ be a D-flopping contraction. Since $\dim \operatorname{Bs} \mathcal{H}_X \leq 0$ and since $\mathcal{H}_X \subseteq |-K_X|$, we see that $-K_X$ is f-nef. Since $\rho(X/Z) = 1$, $-K_X$ is either f-ample or f-trivial. By (2.5), X has only Gorenstein terminal singularities, hence it follows from [14] and [3] that $-K_X$ is f-trivial. Therefore f is a D-flopping contraction with respect to K_X. Thus, by [6] or [10], the D-flop of $(X, \mathcal{H}_X)$ exists and a sequence of D-flops for $(X, \mathcal{H}_X)$ terminates. □

Thus we proved the existence and the termination of D-flops for a terminal pair $(X, \mathcal{H}_X)$ with $\mathcal{H}_X \subseteq |-K_X|$. In order to study the canonical pair $(X, \mathcal{H}_X)$ by the crepant descent method, we need a terminalization of $(X, \mathcal{H}_X)$ and a step to construct a terminalization of $(X, \mathcal{H}_X)$. In the following (2.7), we shall use the results in Section 4.

Theorem 2.7. *Let $(X, \mathcal{H}_X)$ be a canonical pair with $\mathcal{H}_X \subseteq |-K_X|$, and assume that this is not terminal. Then there is a canonical pair $(Y, \mathcal{H}_Y)$ with $\mathcal{H}_Y \subseteq |-K_Y|$ and a projective birational morphism $q : Y \to X$ such that*

(i) *$K_Y + \mathcal{H}_Y = q^*(K_X + \mathcal{H}_X)$, and*

(ii) *$E = \operatorname{Exc}(q)$ is a prime divisor.*

In particular, $e(Y, \mathcal{H}_Y) = e(X, \mathcal{H}_X) - 1$ and $\rho(Y/X) = 1$.

Proof. We shall prove this by dividing into several cases.

(1) If X has non-terminal singularities, then by [6, 6.2], [10, 6.4] or [11, 6.10], there is a projective birational morphism $q : Y \to X$ such that $K_Y = q^*K_X$ and that $E = \operatorname{Exc}(q)$ is a prime divisor. Let $\mathcal{H}_Y = q_*^{-1}\mathcal{H}_X$. We can write $\mathcal{H}_Y = q^*\mathcal{H}_X - aE$ for some $0 \leq a \in \mathbf{Q}$. Since $K_X + \mathcal{H}_X$ is canonical, we see that $a = 0$. Therefore we get $K_Y + \mathcal{H}_Y = q^*(K_X + \mathcal{H}_X)$ in this case.

In the following, we shall assume that X has only terminal singularities. We may also assume that

$\mathrm{Bs}\,\mathcal{H}_X \neq \emptyset$, since otherwise $(X, \mathcal{H}_X)$ is terminal.

(2) If $\mathrm{Bs}\,\mathcal{H}_X$ contains a singular point P of X, then it follows from (4.1) and (4.2) that there is a projective birational morphism $q : Y \to X$ such that $E = \mathrm{Exc}(q)$ is a prime divisor and that $K_Y = q^*K_X + (1/r)E$, where r is the index of X at P. Let $\mathcal{H}_Y = q_*^{-1}\mathcal{H}_X$. Then we see that $\mathcal{H}_Y = q^*\mathcal{H}_X - aE$ for some $1/r \le a \in \mathbf{Q}$. Since $K_X + \mathcal{H}_X$ is canonical, we know that $a = 1/r$ and we get $K_Y + \mathcal{H}_Y = q^*(K_X + \mathcal{H}_X)$.

Thus we may assume that X is nonsingular at each point of $\mathrm{Bs}\,\mathcal{H}_X$.

(3) First we assume that $\mathrm{Bs}\,\mathcal{H}_X$ contains a point P with $\mathrm{mult}_P\,\mathcal{H}_X \ge 2$. In this case, let $q : Y \to X$ be the usual blow up at P and let $\mathcal{H}_Y = q_*^{-1}\mathcal{H}_X$. Then we can argue as in the case (2) and get $K_Y + \mathcal{H}_Y = q^*(K_X + \mathcal{H}_X)$. We also see that $\mathrm{mult}_P\,\mathcal{H}_X = 2$. Otherwise, every point $P \in \mathrm{Bs}\,\mathcal{H}_X$ satisfies $\mathrm{mult}_P\,\mathcal{H}_X = 1$, hence a general member of $\mathcal{H}_X$ is nonsingular at each point of $\mathrm{Bs}\,\mathcal{H}_X$. It follows from (2.5) that $\mathrm{Bs}\,\mathcal{H}_X$ contains an irreducible curve C. In this case, the blow up $q : Y \to X$ along C satisfies our requirement. □

Corollary 2.8. *Let $(X, \mathcal{H}_X)$ be a canonical pair with $\mathcal{H}_X \subseteq |-K_X|$. Then there is a terminal pair $(W, \mathcal{H}_W)$ with $\mathcal{H}_W \subseteq |-K_W|$ and a projective birational morphism $\varphi : W \to X$ such that $K_W + \mathcal{H}_W = \varphi^*(K_X + \mathcal{H}_X)$.*

Proof. This follows from (2.7) by induction on $e(X, \mathcal{H}_X)$. □

Remark 2.9. Let X be a 3-fold with only canonical singularities, and assume that X has non-terminal singularities. Then it follows from [17, 3.7] that there is a projective birational morphism $q : Y \to X$ such that $K_Y = q^*K_X$ and that $\mathrm{Exc}(q)$ contains a divisor (which is not necessarily prime). The construction of $q : Y \to X$ in [17, 3.7] (see also [16, 2.11] and [17, 2.6]) is very concrete, and our terminalization process in (2.7) and (2.8) becomes more explicit if we can take the above $q : Y \to X$ so that $\mathrm{Exc}(q)$ is a prime divisor.

Now we are ready to use the crepant descent method. In what follows, the argument is the same as in [11, Chap. 6].

Theorem 2.10. *Let e be a nonnegative integer. Let $(X, \mathcal{H}_X)$ be a canonical pair such that $\mathcal{H}_X \subseteq |-K_X|$ and $e(X, \mathcal{H}_X) = e$. Let D be a Weil divisor on X. Then there is an integer $m \in [1, 3^{2^e-1}]$ such that mD is Cartier.*

Proof. We shall show this by induction on e. If $e = 0$, then X has only Gorenstein terminal singularities by (2.5). Since X is $\mathbf{Q}$-factorial, every Weil divisor on X is Cartier. Let $e > 0$. By (2.7), there is a canonical pair $(Y, \mathcal{H}_Y)$ with $\mathcal{H}_Y \subseteq |-K_Y|$ and a projective birational morphism $q : Y \to X$ such that $e(Y, \mathcal{H}_Y) = e - 1$. We denote the exceptional divisor of q by E. By [6, 6.8], [7, Step 2] or [11, 6.13], we see that there is a rational curve C on Y such that $q(C)$ is a point and $-3 \le (K_Y + E \cdot C) < 0$. Since Y has only canonical singularities, we can write $K_Y = q^*K_X + \alpha E$ with $0 \le \alpha \in \mathbf{Q}$. Then we get $-3 \le -3/(\alpha+1) = (E \cdot C) < 0$.

Let D be a Weil divisor on X and let D' be its birational transform on Y. By induction hypothesis, there are $m_1, m_2 \in [1, 3^{2^{e-1}-1}] \cap \mathbf{Z}$ such that both m_1E and m_2D' are Cartier. Then the divisor $-(m_1E \cdot C)m_2D' + (m_2D' \cdot C)m_1E$ is Cartier and q-trivial. By [9, 3-2-5], we see that $-(m_1E \cdot C)m_2D$ is a Cartier divisor on X. Since $0 < -(m_1E \cdot C)m_2 \le 3m_1m_2 \le 3^{2^e-1}$, we complete the proof. □

Theorem 2.11. *Let $(X, \mathcal{H}_X)$ be a canonical pair such that $\mathcal{H}_X \subseteq |-K_X|$. Then we have the following*:

(1) *For any effective $\mathbf{Q}$-divisor D on X, the D-flop of $(X, \mathcal{H}_X)$ exists.*

(2) *For any effective $\mathbf{Q}$-divisor D on X, a sequence of D-flops for $(X, \mathcal{H}_X)$ terminates.*

Proof. We shall prove these by induction on $e(X, \mathcal{H}_X)$. If $e(X, \mathcal{H}_X) = 0$, then (1) and (2) are already proved in (2.6).

We shall assume that $e = e(X, \mathcal{H}_X) \ge 1$. Let $f : X \to Z$ be a D-flopping contraction. By (2.7), there is a canonical pair $(Y, \mathcal{H}_Y)$ and a projective birational morphism $q : Y \to X$ such that $K_Y + \mathcal{H}_Y = q^*(K_X + \mathcal{H}_X)$ and that $E = \mathrm{Exc}(q)$ is a prime divisor. Since $\rho(Y/Z) = 2$, $\overline{NE}(Y/Z)$ has two edges Q and R. One of these edges, say Q, is the extremal ray which defines the morphism q. We shall study another edge R. For a sufficiently small positive rational number ε, $(K_Y + \mathcal{H}_Y + \varepsilon q^*D \cdot R) < 0$. By [9, 3-2-1], there is a projective birational morphism $r : Y \to V$ such that $\rho(Y/V) = 1$ and $-q^*D$ is r-ample.

If $\dim \mathrm{Exc}(r) = 1$, then r is a q^*D-flopping contraction. Since $e(Y, \mathcal{H}_Y) = e(X, \mathcal{H}_X) - 1$, we can use the induction hypothesis to $(Y, \mathcal{H}_Y)$. The q^*D-flop of $(Y, \mathcal{H}_Y)$ exists and a sequence of q^*D-flops terminates. Thus there is a sequence of q^*D-flops

$$Y = Y^{(0)} \dashrightarrow Y^{(1)} \dashrightarrow Y^{(2)} \dashrightarrow \cdots \dashrightarrow Y^{(m)},$$

and we may assume that this sequence terminates with $(Y^{(m)}, \mathcal{H}_{Y^{(m)}})$. Let $D^{(m)}$ be the birational transform of q^*D on $Y^{(m)}$. If $D^{(m)}$ is nef over Z, then $|lD^{(m)}|$ is free for some positive integer l, hence defines a projective birational morphism $q_1 : Y^{(m)} \to X_1$ such that $E_1 = \mathrm{Exc}(q_1)$ is a prime divisor. If $D^{(m)}$ is not nef over Z, then the unique extremal ray of $\overline{NE}(Y^{(m)}/Z)$ with respect to $K_{Y^{(m)}} + \mathcal{H}_{Y^{(m)}} + \varepsilon D^{(m)}$ is of divisorial type. Hence there is a projective birational morphism $q_1 : Y^{(m)} \to X_1$ such that $E_1 = \mathrm{Exc}(q_1)$ is a prime divisor. In the case $\dim \mathrm{Exc}(r) = 2$, we set $m = 0$, $q_1 = r$ and $X_1 = V$.

By [11, 6.4, 6.5], we know that X_1 and X are not isomorphic over Z and that $X_1 \to Z$ is the D-flop of $f : X \to Z$. We remark that $D^{(m)}$ is q_1-seminegative in both cases.

Next we shall prove the termination of D-flops for $(X, \mathcal{H}_X)$. Let

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots$$

be an infinite sequence of D-flops. We shall denote the birational transform of D on X_i by D_i. By our construction of D-flops, we have the following diagram:

$$\begin{array}{ccccccc} Y = Y_0 & \dashrightarrow & Y_1 & \dashrightarrow & Y_2 & \dashrightarrow & \cdots \\ \downarrow{\scriptstyle q=q_0} & & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} & & \\ X = X_0 & \dashrightarrow & X_1 & \dashrightarrow & X_2 & \dashrightarrow & \cdots \end{array}$$

where $Y_i \dashrightarrow Y_{i+1}$ is a finite sequence of $q_i^*D_i$-flops, $q_i : Y_i \to X_i$ is a projective birational morphism such that $\rho(Y_i/X_i) = 1$ and that $E_i = \mathrm{Exc}(q_i)$ is a prime divisor. Let D_i' be the birational transform of $q_{i-1}^*D_{i-1}$ on Y_i. Since D_i' is q_i-seminegative, we have $D_i' = q_i^*D_i + \alpha_i E_i$ for some $0 \le \alpha_i \in \mathbf{Q}$.

Here we remark that $e(X_i, \mathcal{H}_{X_i}) = e$ and $e(Y_i, \mathcal{H}_{Y_i}) = e - 1$ for all i. It follows from (2.10) that there is a positive integer $M = M(e)$ such that MD_i' and MD_i are Cartier divisors for all i. Hence $M\alpha_i \in \mathbf{Z}$ for all i.

If $\alpha_i \neq 0$ for infinitely many i, then the divisor $q_i^*D_i$ cannot be effective for $i >> 0$, which is a contradiction. Thus there is a positive integer N such that $\alpha_i = 0$ for all $i \ge N$. Then the sequence $Y_N \dashrightarrow Y_{N+1} \dashrightarrow Y_{N+2} \dashrightarrow \cdots$ gives an infinite sequence of $q_N^*D_N$-flops for $(Y_N, \mathcal{H}_{Y_N})$, which is impossible by induction hypothesis. This completes the proof of the termination of D-flops. $\square$

3. Flips in dimension three. Let X be a normal $\mathbf{Q}$-factorial 3-fold with only terminal singularities.

3.1. A projective birational morphism $f : X \to Z$ onto a normal 3-fold Z is called a *flipping contraction* if $\rho(X/Z) = 1$, $\dim \mathrm{Exc}(f) = 1$ and $-K_X$ is f-ample.

Let $f : X \to Z$ be a flipping contraction. A projective birational morphism $f^+ : X^+ \to Z$ from a normal 3-fold X^+ is called a *flip* of f if $\rho(X^+/Z) = 1$, $\dim \mathrm{Exc}(f^+) = 1$ and K_{X^+} is f^+-ample. It is easy to see that such a morphism $f^+ : X^+ \to Z$ is unique if it exists. The birational map $X \dashrightarrow X^+$ is also called a flip.

3.2. Let $f : X \to Z$ be a flipping contraction. First we shall work in the analytic category. In this case, if we can prove the existence of a flip $f^+ : X^+ \to Z$ under the extra assumption that $\mathrm{Exc}(f)$ is an irreducible curve, then flips exist in general ([6, 8.4], [15, 0.4.3]). Under this assumption, [12, 2.1] proved that a general member S of $|-K_X|$ has only rational double points. In the algebraic category, we cannot use such reductions. However, Mori (unpublished) proved that a general member of $|-K_X|$ has only rational double points even in the case $\mathrm{Exc}(f)$ is reducible.

Theorem 3.3. *Let $f : X \to Z$ be a flipping contraction. Then the flip $f^+ : X^+ \to Z$ of f exists.*

Proof. By (3.2), a general member S of $|-K_X|$ has only rational double points. Thus we obtain a movable linear system $\mathcal{H}_X \subseteq |-K_X|$ which contains S as a general member. By [1, 1.21], we know that the pair $(X, \mathcal{H}_X)$ is canonical. Let m be a positive integer and let $D \in |mK_X|$. Then $-D$ is f-ample and the D-flop $f^+ : X^+ \to Z$ of $(X, \mathcal{H}_X)$ exists by (2.11). Since the birational transform D^+ of D on X^+ is f^+-ample, $f^+ : X^+ \to Z$ is the flip of $f : X \to Z$. $\square$

4. Terminal singularities in dimension three. In this section, we shall study germs of 3-dimensional terminal singularities, and the purpose is to extract exactly one exceptional prime divisor with minimal discrepancy. The results here are used to construct a terminalization of $(X, \mathcal{H}_X)$ in (2.7).

Similar computations are given in [8] and [13], but we need a slightly refined results. We already treat the index ≥ 2 case in [4] and [5] and get more detailed results than the one given here. We shall take a more simple approach which is appropriate for our purpose.

For a polynomial or a power series φ and an integer k, $\mathrm{ord}\,\varphi$ denotes the order of zero at the origin and φ_k denotes the degree k part of φ.

Theorem 4.1. *Let $P \in X$ be a germ of a 3-dimensional Gorenstein terminal singularity, and assume that $P \in X$ is not smooth. Then there is a normal 3-fold Y and a projective birational morphism $q : Y \to X$ such that the exceptional locus $E = \mathrm{Exc}(q)$ is a prime divisor and that the discrepancy $a(E, X) = 1$.*

Proof. Since $P \in X$ is an isolated cDV point, we can embed X into $\mathbf{C}^4$ as a hypersurface.

(1) If $P \in X$ is of type (cA_n), $n \geq 1$, then $X = \{xy + f(z, u) = 0\}$. Let $k = \mathrm{ord}\, f(z, u)$. We take positive integers a, b such that $a + b = k$. Let $q : Y \to X$ be the blow up with weight $(x, y, z, u) = (a, b, 1, 1)$. Then $E \simeq \{xy + f_k(z, u) = 0\} \subseteq \mathbf{P}(a, b, 1, 1)$, which is a prime divisor. It is also easy to see that $a(E, X) = 1$.

(2) If $P \in X$ is of type (cD_n), $n \geq 4$, then $X = \{u^2 + f(x, y, z) = 0\}$, where $\mathrm{ord}\, f(x, y, z) = 3$ and $f_3(x, y, z)$ is not a cube of a linear polynomial. If $f_3(x, y, z)$ is irreducible, then it is sufficient to take the blow up $q : Y \to X$ with weight $(x, y, z, u) = (1, 1, 1, 2)$. Otherwise, by a linear change of coordinates, we may assume that $z \mid f_3(x, y, z)$ and $z^2 \nmid f_3(x, y, z)$. Then the blow up with weight $(x, y, z, u) = (1, 1, 2, 2)$ satisfies our requirement.

(3) If $P \in X$ is of type (cE_n), $n = 6$, 7 or 8, then $X = \{u^2 + x^3 + g(y, z)x + h(y, z) = 0\}$, where $\mathrm{ord}\, g(y, z) \geq 3$ and $\mathrm{ord}\, h(y, z) \geq 4$.

We first treat (cE_7) and (cE_8). In these cases, we have $h_4(y, z) = 0$ and either $g_3(y, z) \neq 0$ or $h_5(y, z) \neq 0$. If $g_3(y, z)$ and $h_5(y, z)$ does not have a common factor, then we take the blow up $q : Y \to X$ with weight $(x, y, z, u) = (2, 1, 1, 3)$. Otherwise, we may assume that $z \mid g_3(y, z)$ and $z \mid h_5(y, z)$. Then we take the blow up with weight $(x, y, z, u) = (2, 1, 2, 3)$.

Lastly, we treat (cE_6). In this case, we have $h_4(y, z) \neq 0$. If $h_4(y, z)$ is not a square, then we take the blow up $q : Y \to X$ with weight $(x, y, z, u) = (2, 1, 1, 2)$. Otherwise, we may assume that $h_4(y, z) = y^2z^2$ or z^4.

In the case $h_4(y, z) = y^2z^2$, we can change the embedding and get $X = \{u^2 + yzu + x^3 + g(y, z)x + h(y, z) = 0\}$, where $\mathrm{ord}\, g(y, z) \geq 3$ and $\mathrm{ord}\, h(y, z) \geq 5$. If "$y \nmid g_3(y, z)$ or $y \nmid h_5(y, z)$" and "$z \nmid g_3(y, z)$ or $z \nmid h_5(y, z)$", then we may take the blow up with weight $(x, y, z, u) = (2, 1, 1, 3)$. Otherwise, we may assume that $z \mid g_3(y, z)$ and $z \mid h_5(y, z)$. Then we take the blow up with weight $(x, y, z, u) = (2, 1, 2, 3)$.

In the case $h_4(y, z) = z^4$, we again can change the embedding and get $X = \{u^2 + z^2u + x^3 + g(y, z)x + h(y, z) = 0\}$, where $\mathrm{ord}\, g(y, z) \geq 3$ and $\mathrm{ord}\, h(y, z) \geq 5$. If $z \nmid g_3(y, z)$ or $z \nmid h_5(y, z)$, then we take the blow up $q : Y \to X$ with weight $(x, y, z, u) = (2, 1, 1, 3)$. Otherwise, we take the blow up with weight $(x, y, z, u) = (2, 1, 2, 3)$. □

Next, we treat the index ≥ 2 case.

Theorem 4.2. *Let $P \in X$ be a germ of a 3-dimensional terminal singularity of index $r \geq 2$. Then there is a normal 3-fold Y and a projective birational morphism $q : Y \to X$ such that the exceptional locus $E = \mathrm{Exc}(q)$ is a prime divisor and that the discrepancy $a(E, X) = 1/r$.*

Proof. We shall prove this by dividing into several cases.

(1) If $P \in X$ is of type (cA/r), ($\mathrm{cAx}/4$) or ($\mathrm{cAx}/2$), then it follows from [4, 6.4, 7.4, 7.8, 8.4, 8.8] that there is a blow up $q : Y \to X$ with required properties.

(2) If $P \in X$ is of type ($\mathrm{cD}/3$), then $X = \{u^2 + f(x, y, z) = 0\}/\mathbf{Z}_3(2, 1, 1, 0)$, where $\mathrm{ord}\, f(x, y, z) = 3$, and we may assume that $f_3(x, y, z) = x^3 + y^2z + yz^2$, $x^3 + yz^2$ or $x^3 + z^3$. Let $q : Y \to X$ be the blow up with weight $(x, y, z, u) = (2/3, 1/3, 4/3, 1)$. Then the exceptional locus E of q is a prime divisor and satisfies $a(E, X) = 1/3$.

(3) If $P \in X$ is of type ($\mathrm{cD}/2$), then $X = \{u^2 + f(x, y, z) = 0\}/\mathbf{Z}_2(1, 1, 0, 1)$, where $\mathrm{ord}\, f(x, y, z) = 3$, and we may assume that $f_3(x, y, z) = xyz$, $xyz + z^3$, y^2z or $y^2z + z^3$. We first assume that $f_4(x, y, 0) = 0$. Then we take the blow up $q : Y \to X$ with weight $(x, y, z, u) = (1/2, 1/2, 2, 3/2)$. Hence we shall assume that $f_4(x, y, 0) \neq 0$ in the following.

In the case $f_3(x, y, z) = xyz$ or $xyz + z^3$, we shall discuss as follows: If $x \nmid f_4(x, y, 0)$ and $y \nmid f_4(x, y, 0)$, then we take the blow up with weight $(x, y, z, u) = (1/2, 1/2, 1, 3/2)$. Otherwise, we may assume that $y \mid f_4(x, y, 0)$, then we take the blow up with weight $(x, y, z, u) = (1/2, 3/2, 1, 3/2)$.

The argument in the case $f_3(x, y, z) = y^2z$ or $y^2z + z^3$ are almost similar. If $y \nmid f_4(x, y, 0)$, then we take the blow up with weight $(x, y, z, u) = (1/2, 1/2, 1, 3/2)$. If $y^2 \mid f_4(x, y, 0)$, then we can reduce to the case $f_4(x, y, 0) = 0$ by changing coordinates. Otherwise, we have $y \parallel f_4(x, y, 0)$. In this case, we take the blow up with weight $(x, y, z, u) = (1/2, 3/2, 1, 3/2)$.

(4) If $P \in X$ is of type ($\mathrm{cE}/2$), then we can write $X = \{u^2 + x^3 + g(y, z)x + h(y, z) = 0\}/\mathbf{Z}_2(0, 1, 1, 1)$, where $\mathrm{ord}\, g(y, z) \geq 4$ and $\mathrm{ord}\, h(y, z) = 4$. We may also assume that $z \mid h_4(y, z)$. Then

it is sufficient to take the blow up with weight $(x, y, z, u) = (1, 1/2, 3/2, 3/2)$. $\square$

Remark 4.3. (1) By [4] and [5], we can choose $q : Y \to X$ in (4.2) so that Y has only terminal singularities. We think that this is also true in the Gorenstein terminal case. If this is true, the terminalization process in (2.7) and (2.8) becomes more explicit.

(2) By using (2.7), we see that Y in (4.1) and (4.2) has only canonical singularities.

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