Determination up to isomorphism of right-angled Coxeter systems

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Abstract: In this paper, we announce that every right-angled Coxeter group determines its Coxeter system up to isomorphism. This implies that the Dranishnikov's rigidity conjecture is the case for right-angled Coxeter groups, i.e., every right-angled Coxeter group determines its boundary up to homeomorphism.

Key words: Coxeter groups; right-angled Coxeter groups; boundaries of groups.

1. Introduction. A *Coxeter group* is a group W having a presentation

$$\langle\, S \mid (st)^{m(s,t)} = 1 \quad \text{for} \quad s,t \in S \,\rangle,$$

where S is a finite set and $m : S \times S \to \mathbf{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(1) $m(s,t) = m(t,s)$ for each $s,t \in S$,
(2) $m(s,s) = 1$ for each $s \in S$, and
(3) $m(s,t) \geq 2$ for each $s,t \in S$ such that $s \neq t$.

The pair (W,S) is called a *Coxeter system*. If, in addition,

(4) $m(s,t) = 2$ or ∞ for each $s,t \in S$ such that $s \neq t$,

then (W,S) is said to be *right-angled*. A group W is called a *right-angled Coxeter group*, if there exists a generating set $S \subset W$ such that (W,S) is a right-angled Coxeter system.

Let (W,S) and (W',S') be Coxeter systems. Two Coxeter systems (W,S) and (W',S') are said to be *isomorphic*, if there exists a bijection $\psi : S \to S'$ such that

$$m(s,t) = m'(\psi(s),\psi(t))$$

for each $s,t \in S$, where $m(s,t)$ and $m'(s',t')$ are the orders of st in W and $s't'$ in W', respectively.

In general, a Coxeter group does not always determine its Coxeter system up to isomorphism. Indeed there exists a counter-example.

Example ([1, p. 38 Exercise 8]). Let $S = \{s,s'\}$ and let

$$W = \langle\, S \mid s^2 = (s')^2 = (ss')^6 = (s's)^6 = 1 \,\rangle.$$

Then (W,S) is a Coxeter system. On the other hand, for $S' = \{(ss')^3, s', s'(ss')^2\}$, (W,S') is a Coxeter system. Since $|S| = 2$ and $|S'| = 3$, these Coxeter systems (W,S) and (W,S') are not isomorphic.

R. Charney and M. W. Davis [4] showed that if a Coxeter group W is capable of acting effectively, properly and cocompactly on some contractible manifold and if (W,S) and (W,S') are Coxeter systems, then $S' = wSw^{-1}$ for some $w \in W$.

The purpose of this note is to announce the following theorem and to state an outline of the proof. A detailed account will be published elsewhere [9].

Theorem 1. *Every right-angled Coxeter group determines its Coxeter system up to isomorphism.*

This means that if a right-angled Coxeter group W admits Coxeter systems (W,S) and (W,S'), then these Coxeter systems are isomorphic.

From a geometric view point of investigation of Coxeter groups, it is known that every Coxeter system (W,S) defines a CAT(0) geodesic space $\Sigma(W,S)$ called the Davis-Vinberg complex ([6, 7, 11]). Then the visual sphere at infinity $\partial\Sigma(W,S)$ of $\Sigma(W,S)$ is called the *boundary of* (W,S). (Details of CAT(0) spaces and their boundaries are found in [2] and [8].) We already know several relation between algebraic properties of W and topological ones of $\partial\Sigma(W,S)$. The following is an important conjecture of this direction, called the *Dranishnikov's Rigidity Conjecture* concerning the boundary of a Coxeter system.

Rigidity conjecture (Dranishnikov [7]). *Every Coxeter group determines its boundary up to homeomorphism. This means that for a Coxeter group W, if (W,S) and (W,S') are Coxeter systems, then the boundaries $\partial\Sigma(W,S)$ and $\partial\Sigma(W,S')$ are homeomorphic.*

If Coxeter systems (W,S) and (W,S') are iso-

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morphic, then the Davis-Vinberg complexes $\Sigma(W,S)$ and $\Sigma(W,S')$ are isometric, and the boundaries $\partial\Sigma(W,S)$ and $\partial\Sigma(W,S')$ are homeomorphic. Thus Theorem 1 gives a partial answer of the Dranishnikov's Rigidity Conjecture.

Corollary 2. *Every right-angled Coxeter group determines its boundary up to homeomorphism.*

In our proof, we had an important property of right-angled Coxeter groups:

Proposition 3. *The order of each element of a right-angled Coxeter group equals either* 1, 2 *or* ∞.

This implies that every Coxeter system of a right-angled Coxeter group is right-angled.

2. Lemmas on Coxeter groups. In this section, we recall some basic properties of Coxeter groups, and we introduce some results for right-angled Coxeter groups.

Definition. Let (W,S) be a Coxeter system. For a subset $T \subset S$, W_T is defined as the subgroup of W generated by T, and called a *parabolic subgroup*. If T is the empty set, then W_T is the trivial group.

Definition. Let (W,S) be a Coxeter system and $w \in W$. A representation $w = s_1 \cdots s_l$ $(s_i \in S)$ is said to be *reduced*, if $\ell(w) = l$, where $\ell(w)$ is the minimum length of word in S which represents w.

The following lemma is known.

Lemma 4 ([1, 3, 5, 10]). *Let (W,S) be a Coxeter system.*

(i) *Let $w \in W$ and let $w = s_1 \cdots s_l$ be a representation. If $\ell(w) < l$, then $w = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_l$ for some $1 \leq i < j \leq l$.*

(ii) *For each subset $T \subset S$, (W_T, T) is a Coxeter system.*

(iii) *Suppose that (W,S) is right-angled. Then W is finite if and only if $st = ts$ for each $s,t \in S$, i.e., $W \cong (\mathbf{Z}_2)^{|S|}$ (hence $|W| = 2^{|S|}$), where $|S|$ is the cardinal number of S.*

Remark. Lemma 4 (iii) implies that every *finite* right-angled Coxeter group determines its Coxeter system up to isomorphism.

Let W be a finite right-angled Coxeter group. Then there exists a generating set $S \subset W$ such that (W,S) is a right-angled Coxeter system. Let $S' \subset W$ such that (W,S') is a Coxeter system. Since $W \cong (\mathbf{Z}_2)^{|S|}$ by Lemma 4 (iii), for each $w \in W \setminus \{1\}$, the order $o(w)$ of w equals 2. Hence $o(s't') = 2$ for each $s',t' \in S'$ with $s' \neq t'$, i.e., (W,S') is right-angled. By Lemma 4 (iii), $(\mathbf{Z}_2)^{|S|} \cong W \cong (\mathbf{Z}_2)^{|S'|}$. Thus $|S| = |S'|$. Since $o(st) = 2 = o(s't')$ for each $s,t \in S$ with $s \neq t$ and each $s',t' \in S'$ with $s' \neq t'$, (W,S) and (W,S') are isomorphic.

By a consequence of Tits solving the word problem ([3, p. 50]), we obtained the following lemma which plays a key role in the proof of the main result.

Lemma 5. *Let (W,S) be a right-angled Coxeter system, let $w \in W$, let $w = s_1 \cdots s_l$ be a reduced representation and let $t,t' \in S$. If $tw = t(s_1 \cdots s_l)$ is reduced and $twt' = w$, then $t = t'$ and $ts_i = s_i t$ for each $i \in \{1,\dots,l\}$.*

Using this lemma, we proved Proposition 3 which implies the following corollary.

Corollary 6. *If W is a right-angled Coxeter group and if (W,S) is a Coxeter system, then (W,S) is right-angled.*

3. Outline of the proof of Theorem 1. For Coxeter systems (W,S) and (W,S'), if W is right-angled, then these Coxeter systems (W,S) and (W,S') are right-angled by Corollary 6. Thus Theorem 1 follows from the following:

Theorem 7. *Let (W,S) and (W',S') be right-angled Coxeter systems. If the Coxeter groups W and W' are isomorphic, then these Coxeter systems (W,S) and (W',S') are isomorphic.*

Let (W,S) and (W',S') be right-angled Coxeter systems such that W and W' are isomorphic, and let $\phi : W \to W'$ be an isomorphism. Let $\mathcal{S}^f := \{T \subset S \mid W_T \text{ is finite}\}$ and let $\mathcal{S'}^f := \{T' \subset S' \mid W'_{T'} \text{ is finite}\}$. We note that $\mathcal{S}^f$ and $\mathcal{S'}^f$ are partially ordered sets with respect to inclusion. Then we proved the following lemmas by Lemma 5 and some basic properties of Coxeter groups.

Lemma 8. *Let T be a maximal element of $\mathcal{S}^f$ with respect to inclusion. Then there exist $w' \in W'$ and a unique maximal element T' of $\mathcal{S'}^f$ such that $\phi(W_T) = w' W'_{T'} (w')^{-1}$.*

Lemma 9. *Let $T_1,\dots,T_k$ be maximal elements of $\mathcal{S}^f$. By Lemma 8, for each $i \in \{1,\dots,k\}$, there exist $w'_i \in W'$ and a unique maximal element T'_i of $\mathcal{S'}^f$ such that $\phi(W_{T_i}) = w'_i W'_{T'_i} (w'_i)^{-1}$. Then $|T_1 \cap \cdots \cap T_k| = |T'_1 \cap \cdots \cap T'_k|$.*

Using Lemmas 8 and 9, we can prove Theorem 7.

Proof of Theorem 7. Let $\phi : W \to W'$ be an isomorphism and let $\{T_1,\dots,T_m\}$ be the set of maximal elements of $\mathcal{S}^f$ with respect to inclusion. For each $i \in \{1,\dots,m\}$, there exist $w'_i \in W'$ and a unique maximal element $T'_i \in \mathcal{S'}^f$ such that $\phi(W_{T_i}) = w'_i W'_{T'_i} (w'_i)^{-1}$ by Lemma 8.

Now we show that $\{T'_1, \ldots, T'_m\}$ is the set of maximal elements of $\mathcal{S}'^f$. Let T' be a maximal element of $\mathcal{S}'^f$. By Lemma 8, $\phi^{-1}(W'_{T'}) = wW_{T_{i_0}}w^{-1}$ for some $w \in W$ and $i_0 \in \{1, \ldots, m\}$. Then

$$\phi(w)^{-1}W'_{T'}\phi(w) = \phi(W_{T_{i_0}}) = w'_{i_0}W'_{T'_{i_0}}(w'_{i_0})^{-1}.$$

By uniqueness, $T' = T'_{i_0}$. Thus $\{T'_1, \ldots, T'_m\}$ is the set of maximal elements of $\mathcal{S}'^f$.

Let $s \in S$. Since $W_{\{s\}} \cong \mathbf{Z}_2$ is finite, $\{s\} \in \mathcal{S}^f$. Hence $\{s\} \subset T_{j_0}$ for some $j_0 \in \{1, \ldots, m\}$, i.e., $s \in T_{j_0} \subset T_1 \cup \cdots \cup T_m$. Thus

$$S = T_1 \cup \cdots \cup T_m.$$

We also have that

$$S' = T'_1 \cup \cdots \cup T'_m$$

by the same argument. By Lemma 9,

(1) $|T_i| = |T'_i|$ for each $i \in \{1, \ldots, m\}$ and
(2) $|\bigcap_{i \in I} T_i| = |\bigcap_{i \in I} T'_i|$ for each subset $I \subset \{1, \ldots, m\}$.

Hence

$$|S| = |T_1 \cup \cdots \cup T_m| = |T'_1 \cup \cdots \cup T'_m| = |S'|.$$

We define a bijection $\psi : S \to S'$ as follows: Let $S = \{s_1, \ldots, s_p\}$. We first define $\psi(s_1)$ as an element of

$$\bigcap\{T'_i \mid i \in \{1, \ldots, m\} \text{ such that } s_1 \in T_i\}$$

which is nonempty by (2). If $\psi(s_1), \ldots, \psi(s_k)$ are defined, then we define $\psi(s_{k+1})$ as an element of

$$\bigcap\{T'_i \mid i \in \{1, \ldots, m\} \text{ such that } s_{k+1} \in T_i\} \setminus \{\psi(s_1), \ldots, \psi(s_k)\}$$

which is nonempty. By induction, we can define a bijection $\psi : S \to S'$ such that

(1) $\psi(T_i) = T'_i$ for each $i \in \{1, \ldots, m\}$ and
(2) $\psi(\bigcap_{i \in I} T_i) = \bigcap_{i \in I} T'_i$ for each subset $I \subset \{1, \ldots, m\}$.

Then we show that for $s, t \in S$, $st = ts$ if and only if $\psi(s)\psi(t) = \psi(t)\psi(s)$. Suppose that $st = ts$. Since $W_{\{s,t\}} \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ is finite, $\{s, t\} \subset T_{i_0}$ for some $i_0 \in \{1, \ldots, m\}$. Then $\{\psi(s), \psi(t)\} \subset \psi(T_{i_0}) = T'_{i_0} \in \mathcal{S}'^f$, i.e., $W'_{\{\psi(s),\psi(t)\}}$ is finite. This means that $\psi(s)\psi(t) = \psi(t)\psi(s)$, since (W', S') is right-angled. Conversely, if $\psi(s)\psi(t) = \psi(t)\psi(s)$, then $\{\psi(s), \psi(t)\} \subset T'_{j_0}$ for some $j_0 \in \{1, \ldots, m\}$, and $\{s, t\} \subset \psi^{-1}(T'_{j_0}) = T_{j_0} \in \mathcal{S}^f$, i.e., $st = ts$.

For each $s, t \in S$ (or $s, t \in S'$), $st = ts$ if and only if $(st)^2 = 1$, and $st \neq ts$ if and only if $o(st) = \infty$ because (W, S) and (W', S') are right-angled. Hence

$$m(s, t) = m'(\psi(s), \psi(t))$$

for each $s, t \in S$. Therefore the right-angled Coxeter systems (W, S) and (W', S') are isomorphic. □

References

[1] Bourbaki, N.: Groupes et Algebrès de Lie. Chapters IV–VI, Masson, Paris (1981).

[2] Bridson, M. R., and Haefliger, A.: Metric Spaces of Non-positive Curvature. Springer-Verlag, Berlin (1999).

[3] Brown, K. S.: Buildings. Springer-Verlag, Berlin (1980).

[4] Charney, R., and Davis, M. W.: When is a Coxeter system determined by its Coxeter group? J. London Math. Soc., **61** (2), 441–461 (2000).

[5] Davis, M. W.: Groups generated by reflections and aspherical manifolds not covered by Euclidean space. Ann. of Math. (2), **117**, 293–324 (1983).

[6] Davis, M. W.: Nonpositive curvature and reflection groups. Handbook of Geometric Topology (eds. Daverman, R. J., and Sher, R. B.). North-Holland, Amsterdam, pp. 373–422 (2002).

[7] Dranishnikov, A. N.: On boundaries of hyperbolic Coxeter groups. Topology Appl., **110** (1), 29–38 (2001).

[8] Ghys, E., and de la Harpe, P. (eds.): Sur les Groupes Hyperboliques d'après Mikhael Gromov. Progr. Math. vol. 83, Birkhäuser, Boston (1990).

[9] Hosaka, T.: Determination up to isomorphism of right-angled Coxeter systems. (2001). (Preprint).

[10] Humphreys, J. E.: Reflection groups and Coxeter groups. Cambridge Univ. Press, Cambridge-New York (1990).

[11] Moussong, G.: Hyperbolic Coxeter groups. Ph.D. Thesis, The Ohio State University (1988).