Integral geometry and Hamiltonian volume minimizing property of a totally geodesic Lagrangian torus in $S^2 \times S^2$

By Hiroshi IRIYEH, Hajime ONO, and Takashi SAKAI

Department of Mathematics, Graduate School of Science, Tokyo Metropolitan University
1-1, Minami-Ohsawa, Hachioji, Tokyo 192-0397
(Communicated by Heisuke HIRONAKA, M. J. A., Dec. 12, 2003)

Abstract: We prove that the product of equators $S^1 \times S^1$ in $S^2 \times S^2$ is globally volume minimizing under Hamiltonian deformations.

Key words: Lagrangian submanifold; Poincaré formula; Hamiltonian stability.

1. Introduction and main results.

In 1990, Y.-G. Oh [4] introduced the notion of *global Hamiltonian stability* of minimal Lagrangian submanifolds in a Kähler manifold and posed the following conjecture:

Conjecture (Oh)**.** Let M be a Kähler-Einstein manifold with an involutive anti-holomorphic isometry τ. Suppose that the fixed point set of τ

$$L := \operatorname{Fix}\tau$$

is also a compact Einstein manifold with positive Ricci curvature. Then for any Hamiltonian isotopy $\rho \in \operatorname{Ham}(M)$ of M, we have

$$\operatorname{vol}(\rho(L)) \geq \operatorname{vol}(L).$$

Kleiner and Oh [4] proved that this conjecture is true for the case $\mathbf{R}P^n \subset \mathbf{C}P^n$ (see also [1]).

Theorem 1 (Kleiner-Oh)**.** *The standard $\mathbf{R}P^n \subset \mathbf{C}P^n$ has the least volume among all its images under Hamiltonian isotopies.*

This is the only known example such that the conjecture has been proved affirmatively.

Important examples of Kähler-Einstein manifolds admitting an involutive anti-holomorphic isometry are Hermitian symmetric spaces. Let M be a Hermitian symmetric space of compact type and τ be a canonical involution on M. Then

$$L := \operatorname{Fix}\tau$$

is a totally geodesic Lagrangian submanifold in M (which is called a *real form* of M). It is interesting to verify the conjecture for such a pair (M, L).

In this paper, we shall prove that the same statement as the conjecture is true in the case of $(S^2 \times S^2 \cong Q_2(\mathbf{C}), S^1 \times S^1)$ although the Lagrangian surface $S^1 \times S^1$ is *flat*. More precisely,

Theorem 2. *Let $L := S^1 \times S^1$ be a totally geodesic Lagrangian torus in $(S^2 \times S^2, \omega_0 \oplus \omega_0)$, where ω_0 denotes the standard Kähler form of $S^2(1) \cong \mathbf{C}P^1$. Then for any Hamiltonian isotopy $\rho \in \operatorname{Ham}(S^2 \times S^2)$, we have*

$$\operatorname{vol}(\rho(L)) \geq \operatorname{vol}(L).$$

Our proof is based on the following Lagrangian intersection theorem ([7], [5] and [6]) and a new Poincaré formula for Lagrangian surfaces in $S^2 \times S^2$.

Theorem 3 (Oh)**.** *Let (M, ω) be a compact symplectic manifold such that there exists an integrable almost complex structure J for which the triple (M, ω, J) becomes a compact Hermitian symmetric space. Let $L = \operatorname{Fix}\tau$ be the fixed point set of an anti-holomorphic involutive isometry τ on M. Assume that the minimal Maslov number of L is greater than or equal to 2. Then for any Hamiltonian isotopy $\rho \in \operatorname{Ham}(M)$ of M such that L and $\rho(L)$ intersect transversally, the inequality*

$$(1) \qquad \sharp(L \cap \rho(L)) \geq \sum_{i=0}^{\dim L} \operatorname{rank} H_i(L, \mathbf{Z}/2\mathbf{Z})$$

holds.

Since the minimal Maslov number of $S^1 \times S^1 \subset S^2 \times S^2$ is 2, the assumption of the above theorem is satisfied in our case.

Proposition 4. *Let N and L be surfaces of $S^2 \times S^2$. Suppose that N is Lagrangian and L is a product of curves in S^2. Then the following inequality holds*:

2000 Mathematics Subject Classification. Primary 53C40; Secondary 53C65.

$$(2)\qquad 4\pi \operatorname{vol}(N)\operatorname{vol}(L) \le \int_{SO(3)\times SO(3)} \sharp(N\cap gL)d\mu(g) \le 16\operatorname{vol}(N)\operatorname{vol}(L).$$

This formula is interesting in its own right. We remark the equality condition of the inequality (2). The first equality of (2) is fulfilled by, for example, a Lagrangian embedding $S^2 \ni z \mapsto (z,-z) \in S^2 \times S^2$. The second equality of (2) holds if and only if the Lagrangian surface N is also a product of closed curves in S^2.

2. Poincaré formula in Riemannian homogeneous spaces. Here we shall review the generalized Poincaré formula in Riemannian homogeneous spaces obtained by Howard [2].

Let U be a finite dimensional real vector space with an inner product, and V and W vector subspaces of dimensions p and q in U, respectively. Take orthonormal bases $v_1,\dots,v_p$ and $w_1,\dots,w_q$ of V and W, and define

$$\sigma(V,W) = \|v_1\wedge\cdots\wedge v_p\wedge w_1\wedge\cdots\wedge w_q\|,$$

which is the angle between V and W.

Let G be a Lie group and K a closed subgroup of G. We assume that G has a left invariant Riemannian metric which is also invariant under elements of K. This metric induces a G-invariant Riemannian metric on G/K. For x and y in G/K and vector subspaces V in $T_x(G/K)$ and W in $T_y(G/K)$, we define $\sigma_K(V,W)$, the angle between V and W, by

$$\sigma_K(V,W) = \int_K \sigma((dg_x)_o^{-1}V, dk_o^{-1}(dg_y)_o^{-1}W)d\mu_K(k)$$

where g_x and g_y are elements of G such that $g_x o = x$ and $g_y o = y$. Here we denote by o the origin of G/K.

Theorem 5 (Howard)**.** *Let G/K be a Riemannian homogeneous space and assume that G is unimodular. Let N and L be submanifolds of G/K with $\dim N + \dim L \ge \dim(G/K)$. Then*

$$\int_G \operatorname{vol}(N\cap gL)d\mu_G(g) = \int_{N\times L} \sigma_K(T_x^\perp N, T_y^\perp L)d\mu(x,y)$$

holds.

The linear isotropy representation induces an action of K on the Grassmannian manifold $G_p(T_o(G/K))$ consisting of all p dimensional subspaces in the tangent space $T_o(G/K)$ at o in a natural way. Although $\sigma_K(T_x^\perp N, T_y^\perp L)$ is defined as an integral on K, we can consider that it is defined as an integral on an orbit of K-action on the Grassmannian manifold. So $\sigma_K(\cdot,\cdot)$ can be regarded as a function defined on the product of the orbit spaces of such K-actions. In the case where G/K is a real space form, $\sigma_K(T_x^\perp N, T_y^\perp L)$ is constant since K acts transitively on the Grassmannian manifold. This implies that the Poincaré formula is expressed as a constant times of the product of the volumes of N and L. In general, such K-actions are not transitive. However, if we can define an invariant for orbits of this action, which is called an *isotropy invariant*, then using this we can express the Poincaré formula more explicitly. From this point of view, Tasaki [8] introduced the multiple Kähler angle, which is the invariant for the actions of unitary groups.

3. Poincaré formula for Lagrangian surfaces in $S^2\times S^2$. In this section we define isotropy invariants for surfaces in $S^2\times S^2$, and give a concrete expression of the Poincaré formula for its Lagrangian surfaces.

Let G be the identity component of the isometry group of $S^2\times S^2$, that is, $G = SO(3)\times SO(3)$. Then the isotropy group K at $o = (p_1,p_2)$ in $S^2\times S^2$ is isomorphic to $SO(2)\times SO(2)$, and $S^2\times S^2$ is expressed as a coset space G/K. Assume that G is equipped with an invariant metric normalized so that G/K becomes isometric to the product of unit spheres. We decompose the tangent space $T_o(G/K)$ as

$$T_o(G/K) = T_{p_1}(S^2)\oplus T_{p_2}(S^2).$$

We take orthonormal bases $\{e_1,e_2\}$ and $\{e_3,e_4\}$ of $T_{p_1}(S^2)$ and $T_{p_2}(S^2)$, respectively, then a complex structure on $T_o(G/K)$ is given by

$$Je_1 = e_2,\quad Je_2 = -e_1,\quad Je_3 = e_4,\quad Je_4 = -e_3.$$

We consider the oriented 2-plane Grassmannian manifold $\tilde{G}_2(T_o(G/K))$. Take an origin $V_o := \operatorname{span}\{e_1,e_2\}$, and express $\tilde{G}_2(T_o(G/K))$ as a coset space

$$\tilde{G}_2(T_o(G/K)) = SO(4)/(SO(2)\times SO(2)) =: G'/K'.$$

Now we study the K-action on $\tilde{G}_2(T_o(G/K))$, and define isotropy invariants. In this case the actions of K and K' on $\tilde{G}_2(T_o(G/K))$ are equivalent by $\operatorname{Ad}: K\to K'$. Therefore it suffices to consider the orbit space of the isotropy action of $\tilde{G}_2(T_o(G/K))$. It is well known that the orbit space of the isotropy action of a symmetric space of compact type can be identified with a fundamental cell of a maximal torus. Hence we can define the isotropy invariant by a coordinate of a maximal torus. We denote by $\mathfrak{g}'$ and $\mathfrak{k}'$ the Lie algebra of G' and K', respectively. Then we have a canonical orthogonal direct sum decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{m}'$ of $\mathfrak{g}'$, where

$$\mathfrak{m}' = \left\{ \begin{pmatrix} O & X \\ -{}^tX & O \end{pmatrix} \;\middle|\; X \in M_2(\mathbf{R}) \right\}.$$

We take a maximal abelian subspace $\mathfrak{a}'$ of $\mathfrak{m}'$ as follows:

$$\mathfrak{a}' = \left\{ \begin{pmatrix} O & X \\ -{}^tX & O \end{pmatrix} \;\middle|\; X = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \theta_1, \theta_2 \in \mathbf{R} \right\}.$$

Then the set of positive restricted roots of $(\mathfrak{g}', \mathfrak{k}')$ with respect to $\mathfrak{a}'$ is

$$\Delta = \{\theta_1 + \theta_2, \theta_1 - \theta_2\}.$$

So we have a fundamental cell C of $\mathfrak{a}'$:

$$C = \left\{ Y = \begin{pmatrix} O & X \\ -{}^tX & O \end{pmatrix} \;\middle|\; X = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \begin{array}{l} 0 \le \theta_1 + \theta_2 \le \pi \\ 0 \le \theta_1 - \theta_2 \le \pi \end{array} \right\}.$$

Thus the isotropy invariants of this case are given by $\theta_1 + \theta_2$ and $\theta_1 - \theta_2$. It is easy to see that the geometric meaning of $\theta_1 - \theta_2$ is the Kähler angle of 2-dimensional subspace Exp$Y$ of $T_o(G/K)$. On the other hand, there is the other complex structure J' which is defined by

$$J'e_1 = e_2,\ J'e_2 = -e_1,\ J'e_3 = -e_4,\ J'e_4 = e_3$$

on $T_o(G/K)$. We can also check that $\theta_1 + \theta_2$ is the Kähler angle of Exp$Y$ with respect to J'.

We attempt to obtain the explicit expression of the Poincaré formula applying the isotropy invariants which we defined above to Theorem 5. Let N and L be surfaces in $S^2 \times S^2$. We take orthonormal bases $\{u_1, u_2\}$ and $\{v_1, v_2\}$ of $(dg_x)_o^{-1}(T_x^\perp N)$ and $(dg_y)_o^{-1}(T_y^\perp L)$, respectively. By the definition, we have

$$\sigma_K(T_x^\perp N, T_y^\perp L) = \int_K \|u_1 \wedge u_2 \wedge k^{-1}(v_1 \wedge v_2)\| d\mu_K(k).$$

Furthermore, by the Hodge star operator,

$$\sigma_K(T_x^\perp N, T_y^\perp L) = \int_K |\langle u_1' \wedge u_2', k^{-1}(v_1 \wedge v_2)\rangle| d\mu_K(k),$$

where $\{u_1', u_2'\}$ is an orthonormal basis of $(dg_x)_o^{-1}(T_x N)$. We put

$$a = \begin{bmatrix} \cos\phi & \sin\phi & & \\ -\sin\phi & \cos\phi & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

$$b = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \cos\psi & \sin\psi \\ & & -\sin\psi & \cos\psi \end{bmatrix}$$

and $k = b^{-1}a$, then we have

$$(3)\quad \sigma_K(T_x^\perp N, T_y^\perp L) = \int_0^{2\pi}\int_0^{2\pi} |\langle a(u_1' \wedge u_2'), b(v_1 \wedge v_2)\rangle| d\phi\, d\psi.$$

Since without loss of generalities we can assume that $(dg_x)_o^{-1}(T_x^\perp N)$ and $(dg_y)_o^{-1}(T_y^\perp L)$ are in Exp$C$, we can take $\{u_1', u_2'\}$ and $\{v_1, v_2\}$ as follows:

$$(4)\quad \begin{aligned} u_1' &= \sin\theta_1 e_1 + \cos\theta_1 e_3, \\ u_2' &= \sin\theta_2 e_2 + \cos\theta_2 e_4, \end{aligned}$$

$$(5)\quad \begin{aligned} v_1 &= \cos\tau_1 e_1 - \sin\tau_1 e_3, \\ v_2 &= \cos\tau_2 e_2 - \sin\tau_2 e_4, \end{aligned}$$

with isotropy invariants $\theta_1 \pm \theta_2$ and $\tau_1 \pm \tau_2$. So we can express the integration of (3) using θ_1, θ_2, τ_1 and τ_2. It is complicated to express this general form, so we shall show for a special case which is needed to prove our main theorem.

Theorem 6. *Let N and L be Lagrangian surfaces in $S^2 \times S^2$. We assume that L is a product of curves in S^2. Then we have*

$$\int_G \sharp(N \cap gL) d\mu(g) = 4\,\mathrm{vol}(L) \int_N \mathrm{length}(\mathrm{Ellip}(\sin^2\theta_x, \cos^2\theta_x)) d\mu(x),$$

where $2\theta_x - \pi/2$ is the Kähler angle of $T_x^\perp N$ with respect to J' and Ellip(α, β) *denotes an ellipse defined by*

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1.$$

Proof. Since N is a Lagrangian surface, $\theta_1 - \theta_2 = \pi/2$ in (4), so we put

$$\theta_1 = \theta, \quad \theta_2 = \theta - \frac{\pi}{2}.$$

On the other hand, L is Lagrangian with respect to both J and J', that is, $\tau_1 = \pi/2$ and $\tau_2 = 0$ in (5). Therefore from (3) we have

$$\sigma_K(T_x^{\perp}N, T_y^{\perp}L) = \int_0^{2\pi}\int_0^{2\pi} |\cos\phi\cos\psi\cos^2\theta + \sin\phi\sin\psi\sin^2|\theta|d\phi\, d\psi.$$

In [3], Kang calculated this type of integrals directly and expressed it by elliptic functions. But we give here a geometrical simple computation. Now we take a subspace which is given by

$$V = \mathrm{span}\{e_2\wedge e_3, e_2\wedge e_4\}$$

in $\bigwedge^2 \big(T_o(G/K)\big)$. Then $b(v_1\wedge v_2)$ moves on the unit circle in V with the parameter ψ. Let P denote the orthogonal projection from $\bigwedge^2 \big(T_o(G/K)\big)$ to V. From (3) we have

$$\sigma_K(T_x^{\perp}N, T_y^{\perp}L) = \int_0^{2\pi}\int_0^{2\pi} |\langle P(a(u_1'\wedge u_2')), b(v_1\wedge v_2)\rangle| d\phi\, d\psi.$$

Here $P(a(u_1'\wedge u_2'))$ moves with parameter ϕ on the ellipse which defined by

$$\frac{x^2}{\cos^4\theta} + \frac{y^2}{\sin^4\theta} = 1$$

in V. Hence we put

$$r_\phi = \sqrt{\cos^2\phi\cos^4\theta + \sin^2\phi\sin^4\theta},$$

then we have

$$\begin{aligned}\sigma_K(T_x^{\perp}N, T_y^{\perp}L) &= \int_0^{2\pi}\int_0^{2\pi} |r_\phi\cos\psi| d\psi\, d\phi\\ &= \int_0^{2\pi} r_\phi d\phi \int_0^{2\pi} |\cos\psi| d\psi\\ &= 4\ \mathrm{length}(\mathrm{Ellip}(\sin^2\theta, \cos^2\theta)).\end{aligned}$$

Thus we complete the proof of Theorem 6 from Theorem 5. □

Proposition 4 is immediately obtained from Theorem 6.

4. Proof of the main theorem.

Proof of Theorem 2. Let $L := S^1\times S^1$ be a totally geodesic Lagrangian torus in $S^2\times S^2$. Let ρ be a Hamiltonian isotopy of $S^2\times S^2$. By inequalities (1) and (2), we have

$$\begin{aligned}&16\,\mathrm{vol}(\rho(L))\,\mathrm{vol}(L)\\ &\geq \int_{SO(3)\times SO(3)} \sharp(\rho(L)\cap gL) d\mu(g)\\ &= \int_{SO(3)\times SO(3)} \sharp(g^{-1}\circ\rho(L)\cap L) d\mu(g)\\ &\geq \int_{SO(3)\times SO(3)} \sum_{i=0}^{2} \mathrm{rank}\, H_i(L, \mathbf{Z}/2\mathbf{Z}) d\mu(g)\\ &= 4\,\mathrm{vol}(SO(3)\times SO(3)).\end{aligned}$$

Since $\mathrm{vol}(SO(3)) = 8\pi^2$ and $\mathrm{vol}(L) = 4\pi^2$, we have

$$\mathrm{vol}(\rho(L)) \geq 4\pi^2 = \mathrm{vol}(L).$$

□

Acknowledgements. We would like to thank Professor Yong-Geun Oh for some helpful comments on the original version of this paper.

The first author is supported by Grant-in-Aid for JSPS Fellows 08889.

The second author is supported by Grant-in-Aid for JSPS Fellows 08832.

References

[1] Givental, A. B.: The Nonlinear Maslov index. London Math. Soc. Lecture Note Ser., vol. 151, Cambridge Univ. Press, Cambridge, pp. 35–43 (1990).

[2] Howard, R.: The kinematic formula in Riemannian homogeneous spaces. Mem. Amer. Math. Soc., **106** (509), pp. 1–69 (1993).

[3] Kang, H. J.: Integral geometry on product of spheres. Tsukuba J. Math., **26** (1), 61–68 (2002).

[4] Oh, Y.-G.: Second variation and stabilities of minimal lagrangian submanifolds in Kähler manifolds. Invent. Math., **101**, 501–519 (1990).

[5] Oh, Y.-G.: Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I. Comm. Pure Appl. Math., **46**, 949–993 (1993).

[6] Oh, Y.-G.: Addendum to "Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I". Comm. Pure Appl. Math., **48**, 1299–1302 (1995).

[7] Oh, Y.-G.: Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. III: Arnold-Givental conjecture. The Floer Memorial volume, Progr. Math., vol. 133, Birkhäuser, Basel, pp. 555–573 (1995).

[8] Tasaki, H.: Generalization of Kähler angle and integral geometry in complex projective spaces. Steps in differential geometry, Proceedings of colloquium on differential geometry, Debrecen, 2000. Inst. Math. Inform., Debrecen, pp. 349–361 (2001). Available electronically at: http://www.emis.de/proceedings/CDGD2000/