Homogeneity, quasi-homogeneity and differentiability of domains

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(Communicated by Heisuke HIRONAKA, M. J. A., Nov. 12, 2003)

Abstract: In this paper we show that any convex quasi-homogeneous projective domain whose boundary is everywhere twice differentiable except possibly at a finite number of points is homogeneous.

Key words: Quasi-homogeneous; homogeneous; divisible; strictly convex.

1. Introduction. A *quasi-homogeneous* projective domain is an open subset Ω of $\mathbf{RP}^n$ which has a compact subset $K \subset \Omega$ and a subgroup G of $\mathrm{Aut}(\Omega)$ such that $GK = \Omega$, where $\mathrm{Aut}(\Omega)$ is a subgroup of $\mathrm{PGL}(n+1, \mathbf{R})$ consisting of all projective transformations preserving Ω. We say Ω is *homogeneous* if we can choose K as a point.

Quasi-homogeneous projective domains are closely related to compact projectively flat manifolds. A projectively flat manifold M is a manifold which is locally modelled on the projective space with its natural projective geometry, i.e., M admits a cover of coordinate charts into the projective space $\mathbf{RP}^n$ whose coordinate transitions are projective transformations. By an analytic continuation of coordinate maps from its universal covering $\tilde{M}$, we obtain a developing map from $\tilde{M}$ into $\mathbf{RP}^n$ and this map is rigid in the sense that it is determined only by a local data. Therefore the deck transformation action on $\tilde{M}$ induces the holonomy action via the developing map by rigidity. (See [3, 6] and [7], etc for more details). Immediately we can see that the developing image of a compact projectively flat manifold is a quasi-homogeneous domain. Particularly when the developing map is a diffeomorphism onto the developing image Ω, Ω is a *divisible* domain, i.e., its automorphism group contains a cocompact discrete subgroup acting properly and freely. So the study about quasi-homogeneous projective domains may be helpful to understand projectively flat manifolds.

There are many quasi-homogeneous domains which are not homogeneous domains. So we have a natural question "When is a quasi-homogeneous domain homogeneous?".

2000 Mathematics Subject Classification. 52A20, 53C24.

Differentiability of the boundary seems to be related to the homogeneity and quasi-homogeneity of domains. For examples, B. Colbois and P. Verovic [2] proved that if Ω is a strictly convex (in the sense that Hessian is positive definite) domain with C^3 boundary which has a cofinite volume discrete subgroup action, then Ω is an ellipsoid. In [4] we showed that any quasi-homogeneous strictly convex domain Ω in $\mathbf{RP}^n$ has a continuously differentiable boundary and it must be an ellipsoid if $\partial\Omega$ is twice differentiable. Furthermore such quasi-homogeneous strictly convex domain Ω fails to be twice differentiable on a dense subset if it is not an ellipsoid. By Proposition 5.15 of [4], any strictly convex quasi-homogeneous domain Ω which is not an ellipsoid has a discrete automorphism group $\mathrm{Aut}(\Omega)$ and so cannot be homogeneous. It is well known there exist infinitely many such non-homogeneous strictly convex quasi-homogeneous projective domains (see [3, 5, 8]).

For an old result on this problem, Vinberg and Kats [8] showed in 2-dimensional case that if a convex quasi-homogeneous domain Ω does not contain any complete line and the boundary of Ω is everywhere twice differentiable except possibly at a finite number of points, then Ω must be homogeneous, that is, Ω is either an ellipse or a triangle. We will show in this paper that this is true for arbitrary dimensional cases.

2. Preliminaries. The purpose of this section is to present some of the basic materials that we will need, which can be mostly found in [4]. We begin with some basic definitions.

The real projective space $\mathbf{RP}^n$ is the quotient space of $\mathbf{R}^{n+1} \setminus \{0\}$ by the action of $\mathbf{R}^* = \mathbf{R} \setminus \{0\}$. In an affine space, we usually denote the affine subspace generated by a subset A by $\langle A \rangle$. So we will use the same notation for a subset of $\mathbf{RP}^n$, i.e., for each subset B of $\mathbf{RP}^n$ $\langle B\rangle$ means the projectivization of the affine subspace generated by $\pi^{-1}(B)$ in $\mathbf{R}^{n+1}$, where π is the quotient map from $\mathbf{R}^{n+1}\setminus\{0\}$ onto $\mathbf{RP}^n$ and we will call $\langle B\rangle$ the *support* of B.

An open subset Ω of $\mathbf{RP}^n$ is called *convex* if there exists an affine space $H\subset\mathbf{RP}^n$ such that Ω is a convex affine subset of H. For a convex projective domain $\Omega\subset\mathbf{RP}^n$, we define an equivalence relation on $\overline{\Omega}$ as follows:

(1) $x\sim y$ if $x\neq y$ and $\overline{\Omega}$ has an open line segment l containing both x and y.

(2) $x\sim y$ if $x=y$.

An equivalence class with respect to this equivalence relation is called a *face* of Ω. Note that a face is relatively open in its support and $\overline{\Omega}$ is a disjoint union of all faces. A convex domain Ω in $\mathbf{RP}^n$ is called *properly convex* if there is no non-constant projective map of $\mathbf{R}$ into Ω and *strictly convex* if $\partial\Omega$ has no line segment. From this definition we see that any strictly convex domain is a properly convex domain.

It is clear that the intersection of any family of convex sets of an affine space is again convex. Therefore, for any subset A of an affine space H there is a smallest convex set containing A in H, namely, the intersection of all convex sets containing A. This convex set is called the convex set spanned by A, or the convex hull of A, and is denoted by $CH(A)$. For a subset A of $\mathbf{RP}^n$ which is contained in some affine space $H\subset\mathbf{RP}^n$, we can also define the convex hull of A. But there might be many different affine spaces in $\mathbf{RP}^n$ containing A. This implies that $CH(A)$ depends on the choice of an affine space H containing A, but even so they are all projectively equivalent and therefore $CH(A)$ is well-defined up to projective transformations.

Since $\mathrm{PM}(n+1,\mathbf{R})$, which is the projectivization of the group of all $n+1$ by $n+1$ matrices, is a compactification of $\mathrm{PGL}(n+1,\mathbf{R})$, any infinite sequence of non singular projective transformations contains a convergent subsequence. Note that the limit projective transformation may be singular in general. For a singular projective transformation g we will denote the projectivization of the kernel and range of g by $K(g)$ and $R(g)$. Then g maps $\mathbf{RP}^n\setminus K(g)$ onto $R(g)$ and the images of any closed set in $\mathbf{RP}^n\setminus K(g)$ under the convergent sequence g_i, converge uniformly to the image under the limit transformation g of g_i (See [1]).

Now we state a useful lemma whose proof can be found in [4].

Lemma 1. *Let Ω be a quasi-homogeneous properly convex domain in $\mathbf{RP}^n$ and G a subgroup of* $\mathrm{Aut}(\Omega)$ *acting on Ω syndetically. Then for each point $p\in\partial\Omega$, there exists a sequence $\{g_i\}\subset G$ and $x\in\Omega$ such that $g_i(x)$ converges to p. Furthermore for any accumulation point g of g_i, $R(g)$ is the projective subspace generated by the face containing p and satisfies the following:*

$$K(g)\cap\Omega=\emptyset \quad\text{and}\quad K(g)\cap\overline{\Omega}\neq\emptyset.$$

The following definitions are originally introduced by Benzécri in [1] using some complicated terminology.

Definition 2.

(i) Let Ω be a properly convex domain in $\mathbf{RP}^n$ and Ω_1 and Ω_2 convex domains in $\langle\Omega_1\rangle$ and $\langle\Omega_2\rangle$ respectively. Ω is called a *convex sum* of Ω_1 and Ω_2, which will be denoted by $\Omega=\Omega_1+\Omega_2$, if $\langle\Omega_1\rangle\cap\langle\Omega_2\rangle=\emptyset$ and Ω is the union of all open line segments joining points in Ω_1 to points in Ω_2.

(ii) A k-dimensional face F of an n-dimensional convex domain Ω is called *conic* if there exist $n-k$ supporting hyperplanes $H_1,H_2,\dots,H_{n-k}$ such that

$$H_1\gneq H_1\cap H_2\gneq\cdots\gneq H_1\cap\cdots\cap H_{n-k}=\langle F\rangle.$$

The following theorem will be used later.

Theorem 3 (Benzécri)**.** *Let Ω be a properly convex quasi-homogeneous projective domain in $\mathbf{RP}^n$. Then*

(i) *Suppose $\Omega=\Omega_1+\Omega_2$. Then Ω is homogeneous* (*respectively, quasi-homogeneous*) *if and only if Ω_i is homogeneous* (*respectively, quasi-homogeneous*) *for $i=1,2$.*

(ii) *If F is a conic face of Ω then there exists a supplementary properly convex domain B such that $\Omega=F+B$.*

(iii) *Let L be a linear subspace of $\mathbf{RP}^n$ of dimension r such that $L\cap\Omega$ has a conic face F. Then there exists a section which is projectively equivalent to an r-dimensional properly convex domain $F+B$ for some suitable properly convex domain B of dimension $r-(\dim(F)+1)$.*

Here a section means an intersection of a projective subspace and Ω.

From (ii) of the above theorem, we get immediately

Corollary 4. *Any quasi-homogeneous properly convex projective domain Ω whose boundary $\partial\Omega$ has a line segment, that is, Ω is not strictly convex. Then Ω has a triangular section Δ_{abc}.*

Proof. See Corollary 2.9 of [4]. □

3. Quasi-homogeneous convex projective domains. This section is devoted to prove that twice differentiability of $\partial\Omega$ with exception of finite points implies homogeneity of a properly convex quasi-homogeneous projective domain Ω.

There are many n-dimensional polyhedras if $n > 1$, but the following proposition implies that a quasi-homogeneous polyhedron is unique up to projective equivalence.

Proposition 5. *Simplices are the only quasi-homogeneous polyhedras in $\mathbf{RP}^n$.*

Proof. We proceed by induction on $n =$ dimension of polyhedron with the case $n = 1$ being trivial. If $n > 1$ and the proposition is true for all $i < n$, choose any vertex v of n-dimensional polyhedron Ω. Then v is a conic face and thus $\Omega = v + \Omega'$ for some $(n-1)$-dimensional quasi-homogeneous properly convex domain by Theorem 3. Since Ω is an n-dimensional polyhedron, Ω' is also a polyhedron of dimension $(n-1)$. Induction hypothesis implies that Ω' is a $(n-1)$-dimensional simplex. This completes the proof. □

For a quasi-homogeneous projective domain having twice differentiable boundary, we get the following.

Theorem 6. *Let Ω be a properly convex quasi-homogeneous projective domain whose boundary is twice differentiable. Then Ω is an ellipsoid.*

Proof. This is an immediate consequence of Theorem 5.1 and Proposition 5.10 of [4]. □

The above thoerem means that any properly convex quasi-homogeneous projective domain is homogeneous if its boundary is twice differentiable. Then what can we say for convex quasi-homogeneous projective domains whose boundary is not twice differentiable everywhere? Now we can prove the following theorem, which is a generalization of 2-dimensional result of Vinberg and Kats [8]. Since their proof cannot be applied directly in arbitrary dimensional cases, we apply some different technique using the results of Benzécri [1].

Theorem 7. *Let Ω be a properly convex quasi-homogeneous projective domain in $\mathbf{RP}^n$ such that the boundary of Ω is everywhere twice differentiable except possibly at a finite number of points. Then Ω is homogeneous. Furthermore Ω is an ellipsoid if $n \geq 3$ and Ω is either an ellipse or a triangle if $n = 2$ and Ω is an interval if $n = 1$.*

Proof. By Theorem 6, Ω is an ellipsoid if it is twice differentiable everywhere. So we may assume that $S = \{p_1, \ldots, p_k\}$ is the set of all points at which the boundary of Ω is not twice differentiable. Then S and $CH(S)$ are invariant subset.

Suppose that $CH(S)$ is a proper subset of $\overline{\Omega}$. Since $\overline{\Omega}$ is the convex hull of the set of all extreme points of Ω we can take another extreme point q of Ω and so q has a neighborhood on which the boundary of Ω is twice differentiable. By Lemma 1, we can take a sequence $\{g_i\} \subset \mathrm{Aut}(\Omega)$ such that $\lim_{i\to\infty} g_i = g$ and $R(g) = p$ since p is extremal. Suppose $\partial\Omega$ has a line segment. Then Ω has a triangular section Δ by Corollary 4. Since $K(g) \cap \Omega = \emptyset$ by Lemma 1, there is a vertex v of Δ which does not contained in $K(g)$. Notice that $g_n(v)$ converges to p and thus $g_n(v)$ belongs to $U \cap \partial\Omega$ for sufficiently large n. But $\partial\Omega$ is not differentiable at $g_n(v)$ and $U \cap \partial\Omega$ is twice differentiable, which is a contradiction. So a polyhedron $CH(S)$ must be equal to Ω. By Proposition 5, Ω must be a simplex. Obviously, a triangle and an interval are all simplex such that its boundary is twice differentiable with finite point exception. □

Remark 8. The above proof says actually that any quasi-homogeneous properly convex projective domain with an extreme point p having differentiable neighborhood U_p in $\partial\Omega$ is a strictly convex domain, and furthermore if U_p is twice differentiable then Ω is an ellipsoid by Theorem 5.1 in [4].

Corollary 9. *Let Ω be a quasi-homogeneous domain in $\mathbf{RP}^n$. Suppose that the closure of the convex hull $CH(\Omega)$ of Ω is contained in an affine space, that is, $CH(\Omega)$ is properly convex. Then Ω is homogeneous if the boundary of Ω is everywhere twice differentiable except possibly at a finite number of points.*

Proof. It is well known that Ω is properly convex if the closure of the convex hull $CH(\Omega)$ of Ω is contained in an affine space (see Prop. 2 in Section II.3 of [1] or Vinberg and Kats [8]). So the conclusion is follows from the above theorem. □

4. Remarks. For dimension ≥ 3, the assumption in Theorem 7 that $\partial\Omega$ is everywhere twice differentiable *except possibly finite points* seems to be too strong. There must be a weaker condition implying the homogeneity.

In two-dimensional case, it is well known that a properly convex quasi-homogeneous projective domain is either an ellipse, or a triangle, or a strictly convex domain whose boundary is not twice differentiable. But we know that any strictly convex quasi-homogeneous projective domain Ω has a C^1 boundary (see [4]) and we can show that f is twice differentiable almost everywhere and $f''(x) = 0$ for any twice differentiable point x, where f is a local defining function of $\partial\Omega$: if there exists $x \in \partial\Omega$ such that $f''(x) \neq 0$, then $f''(x)$ must be positive by strict convexity of f and thus we can show that it must be an ellipsoid by the same argument with the proof of Theorem 5.1 of [4]. From this classification of 2-dimensional properly convex quasi-homogeneous projective domains, we see that a properly convex quasi-homogeneous projective domain may not be homogeneous even if $\partial\Omega$ is twice differentiable on a dense subset.

But if we strengthen the condition somewhat, it seems to imply the homogeneity. Observe that any n-simplex has infinitely many non-differentiable point for $n > 2$ but *the set of twice differentiable points is an open dense subset*. All other known examples for homogeneous domains have this property.

On the other hand, the proof of Theorem 7 says actually that any quasi-homogeneous properly convex projective domain with an extreme point p having twice differentiable neighborhood U_p in $\partial\Omega$ is homogeneous (see Remark 8). But this fact does not imply that a properly convex quasi-homogeneous projective domain Ω is homogeneous if $\partial\Omega$ is twice differentiable on some neighborhood of a boundary point. An easy example is a strictly convex quasi-homogeneous cone which is not an elliptic cone: Every strictly convex quasi-homogeneous cone has a boundary which is twice differentiable on some neighborhood of a point in their maximal dimensional proper face. But it is not homogeneous if it is not an elliptic cone. We will show in another paper that a quasi-homogeneous convex projective domain is homogeneous if the boundary is twice differentiable on some special kind of neighborhood of a boundary point (it is called a '*pocket*' which is introduced by Benzécri at first).

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