Normalized double sine functions

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(Communicated by Shokichi Iyanaga, m. j. a., Jan. 14, 2003)

Abstract: We express normalized double sine functions of integer periods (N_1, N_2) via the standard double sine function of period $(1, 1)$. As an application we give an Euler product expression using the di-logarithm for the double zeta function $\zeta(s, \mathbf{F}_{p^{N_1}}) \otimes \zeta(s, \mathbf{F}_{p^{N_2}})$ for a prime number p and integers N_1, N_2.

Key words: Zeta functions; sine functions; multiple zeta functions; multiple sine functions.

1. Definitions and results. Normalized multiple sine functions are generalizations of the usual sine function. We studied their basic properties in previous papers [KoKu1, KoKu2, KuKo] with some applications.

For $\omega_1, \ldots, \omega_r > 0$ and $x > 0$, the multiple Hurwitz zeta function is defined by Barnes [B] as

$$\zeta_r(s, x, (\omega_1, \ldots, \omega_r))$$
$$:= \sum_{n_1, \ldots, n_r = 0}^{\infty} (n_1 \omega_1 + \cdots + n_r \omega_r + x)^{-s}$$

in $\mathrm{Re}(s) > r$. This has the analytic continuation to all $s \in \mathbf{C}$ as a meromorphic function, and it is holomorphic at $s = 0$. Then the normalized multiple gamma function is defined as

$$\Gamma_r(x, (\omega_1, \ldots, \omega_r))$$
$$:= \exp\left(\frac{\partial}{\partial s} \zeta_r(s, x, (\omega_1, \ldots, \omega_r)) \Big|_{s=0} \right).$$

This is a constant multiple of the multiple gamma function $\Gamma_r^B(x, (\omega_1, \ldots, \omega_r))$ of Barnes [B]:

$$\Gamma_r(x, (\omega_1, \ldots, \omega_r))$$
$$= \Gamma_r^B(x, (\omega_1, \ldots, \omega_r)) / \rho_r(\omega_1, \ldots, \omega_r).$$

Now, the normalized multiple sine function is

$$S_r(x, (\omega_1, \ldots, \omega_r))$$
$$:= \Gamma_r(x, (\omega_1, \ldots, \omega_r))^{-1}$$
$$\times \Gamma_r(\omega_1 + \cdots + \omega_r - x, (\omega_1, \ldots, \omega_r))^{(-1)^r}.$$

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For example

$$S_1(x, \omega) = \Gamma_1(x, \omega)^{-1} \Gamma_1(\omega - x, \omega)^{-1}$$
$$= 2 \sin(\pi x / \omega),$$

since we have $\Gamma_1(x, \omega) = (2\pi)^{-1/2} \Gamma(x/\omega) \omega^{\frac{x}{\omega} - \frac{1}{2}}$ from $\zeta_1(s, x, \omega) = \omega^{-s} \zeta(s, x/\omega)$.

To simplify the notation we put

$$S_r(x) := S_r(x, (1, \ldots, 1)),$$
$$\Gamma_r(x) := \Gamma_r(x, (1, \ldots, 1)),$$
$$\zeta_r(s, x) := \zeta_r(s, x, (1, \ldots, 1)).$$

Hence

$$S_r(x) = \Gamma_r(x)^{-1} \Gamma_r(r - x)^{(-1)^r}$$

and

$$\Gamma_r(x) = \exp\left(\frac{\partial}{\partial s} \zeta_r(s, x) \Big|_{s=0} \right).$$

Here we investigate normalized double sine functions, especially in the rational period cases: $S_2(x, (\omega_1, \omega_2))$ with $\omega_2/\omega_1 \in \mathbf{Q}$. The following theorem expresses them in terms of $S_2(x)$:

Theorem 1.1. *Let N_1, N_2 be positive integers with the greatest common divisor N_0. Then we have*

$$(1.1) \quad S_2(x, (N_1, N_2))$$
$$= \prod_{k_1=0}^{(N_2/N_0)-1} \prod_{k_2=0}^{(N_1/N_0)-1} S_2\left(\frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2 / N_0} \right).$$

As application of Theorem 1.1, we compute the absolute tensor product of the Hasse zeta functions of finite fields with p^{N_1} and p^{N_2} elements:

Theorem 1.2. *Let N_1, N_2 be positive integers with the greatest common divisor N_0. The absolute tensor product of the Hasse zeta functions for finite fields $\mathbf{F}_{p^{N_1}}$ and $\mathbf{F}_{p^{N_2}}$ is given as follows:*

$$\zeta(s, \mathbf{F}_{p^{N_1}}) \otimes \zeta(s, \mathbf{F}_{p^{N_2}})$$

$$= \exp\left(-\frac{1}{2\pi i}\frac{N_0^2}{N_1 N_2}\sum_{n=1}^{\infty}\frac{p^{-snN_1N_2/N_0}}{n^2}\right.$$

$$+ \left(\frac{isN_0\log p}{2\pi} - 1\right)\sum_{n=1}^{\infty}\frac{p^{-snN_1N_2/N_0}}{n}$$

$$+ \sum_{n=1}^{\infty}\frac{p^{-snN_1}}{n}f_1(n)$$

$$\left.+ \sum_{n=1}^{\infty}\frac{p^{-snN_2}}{n}f_2(n) + Q_p(s)\right),$$

where

$$f_1(n) = \begin{cases} (e^{2\pi i n N_1/N_2} - 1)^{-1} & \left(\frac{N_2}{N_0} \nmid n\right) \\ \dfrac{N_2 - N_0}{2N_0} & \left(\frac{N_2}{N_0} \mid n\right) \end{cases},$$

$$f_2(n) = \begin{cases} (e^{2\pi i n N_2/N_1} - 1)^{-1} & \left(\frac{N_1}{N_0} \nmid n\right) \\ \dfrac{N_1 - N_0}{2N_0} & \left(\frac{N_1}{N_0} \mid n\right) \end{cases}$$

and $Q_p(s)$ is a quadratic polynomial in s.

2. Proof of Theorem 1.1. It suffices to show when $N_0 = 1$, since the homogeneity [KuKo, Theorem 2.1 (e)] of the multiple sine functions gives

$$S_2(x, (N_1, N_2)) = S_2\left(\frac{x}{N_0}, \left(\frac{N_1}{N_0}, \frac{N_2}{N_0}\right)\right).$$

In case $N_0 = 1$, the right hand side of (1.1) is calculated as follows:

$$(2.1) \quad \prod_{k_1=0}^{N_2-1}\prod_{k_2=0}^{N_1-1} S_2\left(\frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right)$$

$$= \prod_{k_1=0}^{N_2-1}\prod_{k_2=0}^{N_1-1} \Gamma_2\left(\frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right)^{-1}$$

$$\Gamma_2\left(2 - \frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right)$$

$$= \prod_{k_1=0}^{N_2-1}\prod_{k_2=0}^{N_1-1} \exp\left(\frac{\partial}{\partial s}\bigg|_{s=0}\right.$$

$$\left(-\zeta_2\left(s, \frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right)\right.$$

$$\left.\left.+ \zeta_2\left(s, 2 - \frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right)\right)\right)$$

$$= \exp\left(\frac{\partial}{\partial s}\bigg|_{s=0}\sum_{k_1=0}^{N_2-1}\sum_{k_2=0}^{N_1-1}\right.$$

$$\left(-\zeta_2\left(s, \frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right)\right.$$

$$\left.\left.+ \zeta_2\left(s, 2 - \frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right)\right)\right).$$

The double sum is computed as follows:

$$\sum_{k_1=0}^{N_2-1}\sum_{k_2=0}^{N_1-1}$$

$$\left(-\sum_{m_1,m_2\geq 0}\left(m_1 + m_2 + \frac{x+N_1k_1+N_2k_2}{N_1N_2}\right)^{-s}\right.$$

$$\left.+ \sum_{m_1,m_2\geq 0}\left(m_1 + m_2 + 2 - \frac{x+N_1k_1+N_2k_2}{N_1N_2}\right)^{-s}\right)$$

$$= (N_1 N_2)^s \sum_{k_1=0}^{N_2-1}\sum_{k_2=0}^{N_1-1}$$

$$\left(-\sum_{m_1,m_2\geq 0}\left((m_1 N_2 + k_1)N_1\right.\right.$$

$$\left.+ (m_2 N_1 + k_2)N_2 + x\right)^{-s}$$

$$+ \sum_{m_1,m_2\geq 0}\left((m_1 N_2 + N_2 - k_1 - 1)N_1\right.$$

$$+ (m_2 N_1 + N_1 - k_2 - 1)N_2$$

$$\left.\left.+ N_1 + N_2 - x\right)^{-s}\right)$$

$$= (N_1 N_2)^s\left(-\zeta_2(s, x, (N_1, N_2))\right.$$

$$\left.+ \zeta_2(s, N_1 + N_2 - x, (N_1, N_2))\right).$$

We previously obtained in the proof of [KuKo, Theorem 2.1(b)] that the function

$$-\zeta_2(s, x, (N_1, N_2)) + \zeta_2(s, N_1 + N_2 - x, (N_1, N_2))$$

has zeros at even nonnegative integers s. In particular it vanishes at $s = 0$, thus (2.1) equals

$$\exp\left(\frac{\partial}{\partial s}\bigg|_{s=0}\left(-\zeta_2(s, x, (N_1, N_2))\right.\right.$$

$$\left.\left.+ \zeta_2(s, N_1 + N_2 - x, (N_1, N_2))\right)\right)$$

$$= \Gamma_2(x, (N_1, N_2))^{-1}\Gamma_2(N_1 + N_2 - x, (N_1, N_2))$$

$$= S_2(x, (N_1, N_2)).$$

This completes the Proof of Theorem 1.1. □

3. Application. In this section we compute the absolute tensor product of the Hasse zeta functions of finite fields $\mathbf{F}_{p^{N_1}}$ and $\mathbf{F}_{p^{N_2}}$. We first recall the definition of the absolute tensor product of meromorphic functions. Let Z_j ($j = 1, 2$) be meromorphic functions of order μ_j. We put the Hadamard product as

$$(3.1) \qquad Z_j(s) = s^{k_j} e^{Q_j(s)} \prod_{\rho \in \mathbf{C}}' P_{\mu_j}\left(\frac{s}{\rho}\right)^{m_j(\rho)},$$

where

$$P_r(u) := (1 - u) \exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r}\right),$$

m_j denotes the multiplicity function with $k_j := m_j(0)$, and Q_j is a polynomial with $\deg Q_j \leq \mu_j$. Here the product over $\rho \in \mathbf{C}$ means $\lim_{R \to \infty} \prod_{0 < |\rho| < R} P_{\mu_j}\left(\frac{s}{\rho}\right)^{m_j(\rho)}$. The absolute tensor product is defined by

$$(3.2) \quad (Z_1 \otimes Z_2)(s)$$
$$:= s^{k_1 k_2} e^{Q(s)} \prod_{\rho_1, \rho_2 \in \mathbf{C}}' P_{\mu_1 + \mu_2}\left(\frac{s}{\rho_1 + \rho_2}\right)^{m(\rho_1, \rho_2)},$$

where $Q(s)$ is a polynomial with $\deg Q \leq \mu_1 + \mu_2$ and

$$m(\rho_1, \rho_2)$$
$$:= m_1(\rho_1) m_2(\rho_2) \times \begin{cases} 1 & \text{if } \operatorname{Im}(\rho_1), \operatorname{Im}(\rho_2) \geq 0, \\ -1 & \text{if } \operatorname{Im}(\rho_1), \operatorname{Im}(\rho_2) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here we do not give the precise definition of the polynomial $Q(s)$, since it is not necessary for our purpose. We refer to Manin [M] for an excellent survey.

In this section we will compute this absolute tensor product for the Hasse zeta functions for finite fields:

$$Z_1(s) = \zeta(s, \mathbf{F}_{p^{N_1}}) = (1 - p^{-N_1 s})^{-1},$$
$$Z_2(s) = \zeta(s, \mathbf{F}_{p^{N_2}}) = (1 - p^{-N_2 s})^{-1},$$

with p a prime number and N_1, N_2 positive integers.

The following proposition extends our previous results on $\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_p)$ for a prime p in [KoKu2].

Proposition 3.1. *The absolute tensor product of the Hasse zeta functions for finite fields $\mathbf{F}_{p^{N_1}}$ and $\mathbf{F}_{p^{N_2}}$ is given as follows:*

$$\zeta(s, \mathbf{F}_{p^{N_1}}) \otimes \zeta(s, \mathbf{F}_{p^{N_2}})$$
$$= e^{Q(s)} S_2\left(\frac{is N_1 N_2 \log p}{2\pi}, (N_1, N_2)\right)$$

$$= e^{Q(s)} \prod_{k_1=0}^{(N_2/N_0)-1} \prod_{k_2=0}^{(N_1/N_0)-1} S_2\left(N_0\left(\frac{is \log p}{2\pi} + \frac{k_1}{N_2} + \frac{k_2}{N_1}\right)\right),$$

where $N_0 = (N_1, N_2)$ and $Q(s)$ is a polynomial of degree at most two, which depends on p.

Proof. The second equality is seen from Theorem 1.1. In what follows we prove the first one. The Hadamard product (3.1) for the Hasse zeta function is given for $j = 1, 2$ by

$$\zeta(s, \mathbf{F}_{p^{N_j}}) = s^{-1} e^{\tilde{Q}_{N_j}(s)} \prod_{n=-\infty}^{\infty}' P_1\left(\frac{s}{\frac{2\pi i}{N_j \log p} n}\right)^{-1}$$

with $\tilde{Q}_{p,j}(s)$ a linear polynomial depending on N_j. Thus by the definition (3.2) of the absolute tensor product,

$$\zeta(s, \mathbf{F}_{p^{N_1}}) \otimes \zeta(s, \mathbf{F}_{p^{N_2}})$$
$$= s e^{\tilde{Q}_{N_1, N_2}(s)} \prod_{k, n \in \mathbf{Z}}' P_2\left(\frac{s}{\frac{2\pi i}{N_1 \log p} k + \frac{2\pi i}{N_2 \log p} n}\right)^{m_{k,n}},$$

where $\tilde{Q}_{N_1, N_2}(s)$ is a polynomial of degree at most two and

$$m_{k,n} := m\left(\frac{2\pi i}{N_1 \log p} k, \frac{2\pi i}{N_2 \log p} n\right)$$
$$= \begin{cases} 1 & \text{if } k, n \geq 0 \\ -1 & \text{if } k, n < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\zeta(s, \mathbf{F}_{p^{N_1}}) \otimes \zeta(s, \mathbf{F}_{p^{N_2}})$$
$$= e^{\tilde{Q}_{N_1, N_2}(s)} s$$
$$\times \frac{\displaystyle\prod_{k, n=0}^{\infty}' P_2\left(\frac{s}{\frac{2\pi i}{N_1 \log p} k + \frac{2\pi i}{N_2 \log p} n}\right)}{\displaystyle\prod_{k, n=1}^{\infty} P_2\left(-\frac{s}{\frac{2\pi i}{N_1 \log p} k + \frac{2\pi i}{N_2 \log p} n}\right)}.$$

We appeal to the $r = 2$ case of the formula [KuKo, Proposition 2.4]:

$$S_2(z, (\omega_1, \omega_2))$$
$$= e^{Q_{\underline{\omega}}(z)} z \frac{\displaystyle\prod_{k, n=0}^{\infty}' P_2\left(-\frac{z}{\omega_1 k + \omega_2 n}\right)}{\displaystyle\prod_{k, n=1}^{\infty} P_2\left(\frac{z}{\omega_1 k + \omega_2 n}\right)}$$

where $Q_{\underline{\omega}}(z)$ a polynomial with $\deg Q_{\underline{\omega}} \leq 2$. The proof is complete. □

Taking the following exponential expression of the normalized double sine functions into account, we see that the absolute tensor product in Theorem 1.2 has an "Euler product" expression for $\mathrm{Re}(s) > 0$.

Proposition 3.2. *The following expression holds for* $\mathrm{Im}\, z > 0$.

$$S_2(z) = \exp\left(-\frac{1}{2\pi i}\mathrm{Li}_2(e^{2\pi iz})\right.$$
$$\left. + (1-z)\log(1-e^{2\pi iz}) + Q(z)\right),$$

where $Q(z) = \frac{\pi i}{2}z^2 - \pi i z + \frac{5\pi i}{12}$.

Proof. We recall the formulas of the double sine functions [KuKo, Example 3.6]:

$$(3.3) \qquad S_2(z) = \mathscr{S}_2(z)^{-1}\mathscr{S}_1(z),$$

where $\mathscr{S}_r(z)$ $(r = 1, 2)$ are the primitive multiple sine functions [KuKo]. We have by definition

$$\mathscr{S}_1(z) = 2\sin \pi z$$
$$= \exp\left(-\pi i z + \frac{\pi i}{2} + \log(1-e^{2\pi iz})\right)$$

and the expression [KuKo, Theorem 2.8 (2.12)]:

$$\mathscr{S}_2(z) = \exp\left(\frac{1}{2\pi i}\mathrm{Li}_2(e^{2\pi iz})\right.$$
$$\left. + z\log(1-e^{2\pi iz}) - \frac{\pi i}{2}z^2 - \frac{\zeta(2)}{2\pi i}\right)$$

for $\mathrm{Im}(z) > 0$. Thus (3.3) equals

$$\exp\left(-\frac{1}{2\pi i}\mathrm{Li}_2(e^{2\pi iz}) + (1-z)\log(1-e^{2\pi iz})\right.$$
$$\left. + \frac{\pi i}{2}z^2 - \pi i z + \frac{5\pi i}{12}\right).$$
□

Lemma 3.3. *Assume* $r \in \mathbf{C}$ *satisfies that* $r^N = 1$. *Then we have*

$$\frac{1}{N}\sum_{n=0}^{N-1} nr^n = \begin{cases} (r-1)^{-1} & (r \neq 1) \\ \dfrac{N-1}{2} & (r = 1). \end{cases}$$

Proof. The $r = 1$ case is well-known. Differentiating the formula $\sum_{n=0}^{N-1} r^n = (1-r^N)/(1-r)$ in case $r \neq 1$ leads to the result. □

Proof of Theorem 1.2. By the above propositions all we should compute is the following product:

$$\prod_{k_1=0}^{(N_2/N_0)-1}\prod_{k_2=0}^{(N_1/N_0)-1} S_2\left(N_0\left(\frac{is\log p}{2\pi} + \frac{k_1}{N_2} + \frac{k_2}{N_1}\right)\right).$$

This is equal to the exponential of the sum of the following double series (3.4), (3.5) and (3.6):

(3.4)

$$-\sum_{k_1=0}^{\frac{N_2}{N_0}-1}\sum_{k_2=0}^{\frac{N_1}{N_0}-1}\frac{1}{2\pi i}\mathrm{Li}_2\left(e^{-N_0 s\log p + 2\pi i\left(\frac{N_0 k_1}{N_2} + \frac{N_0 k_2}{N_1}\right)}\right),$$

(3.5)

$$\sum_{k_1=0}^{\frac{N_2}{N_0}-1}\sum_{k_2=0}^{\frac{N_1}{N_0}-1}\left(1 - \frac{isN_0\log p}{2\pi} - \frac{N_0 k_1}{N_2} - \frac{N_0 k_2}{N_1}\right)$$
$$\log\left(1 - e^{-sN_0\log p + 2\pi i\left(\frac{N_0 k_1}{N_2} + \frac{N_0 k_2}{N_1}\right)}\right),$$

$$(3.6) \quad \sum_{k_1=0}^{\frac{N_2}{N_0}-1}\sum_{k_2=0}^{\frac{N_1}{N_0}-1} Q\left(\frac{isN_0\log p}{2\pi} + \frac{N_0 k_1}{N_2} + \frac{N_0 k_2}{N_1}\right).$$

Put $N_1 = N_0 N_1'$ and $N_2 = N_0 N_2'$. First the double sum (3.4) is computed as follows:

$$-\sum_{k_1=0}^{N_2'-1}\sum_{k_2=0}^{N_1'-1}\frac{1}{2\pi i}\mathrm{Li}_2\left(e^{-N_0 s\log p + 2\pi i\left(\frac{k_1}{N_2'} + \frac{k_2}{N_1'}\right)}\right)$$

$$= -\frac{1}{2\pi i}\sum_{k_1=0}^{N_2'-1}\sum_{k_2=0}^{N_1'-1}\sum_{n=1}^{\infty}\frac{\left(p^{-sN_0}e^{2\pi i\left(\frac{k_1}{N_2'} + \frac{k_2}{N_1'}\right)}\right)^n}{n^2}$$

$$= -\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{p^{-sN_0 n}}{n^2}\sum_{k_1=0}^{N_2'-1}e^{2\pi i\frac{nk_1}{N_2'}}\sum_{k_2=0}^{N_1'-1}e^{2\pi i\frac{nk_2}{N_1'}}$$

$$= -\frac{1}{2\pi i}\sum_{N_1'N_2'|n}\frac{p^{-sN_0 n}}{n^2}N_1'N_2'$$

$$= -\frac{1}{2\pi i}\frac{1}{N_1'N_2'}\sum_{n=1}^{\infty}\frac{p^{-sN_0 N_1'N_2' n}}{n^2}$$

$$= -\frac{1}{2\pi i}\frac{N_0^2}{N_1 N_2}\sum_{n=1}^{\infty}\frac{p^{-snN_1 N_2/N_0}}{n^2}.$$

Next we compute the double sum of (3.5):

$$\sum_{k_1=0}^{N_2'-1}\sum_{k_2=0}^{N_1'-1}\left(1 - \frac{isN_0\log p}{2\pi} - \frac{k_1}{N_2'} - \frac{k_2}{N_1'}\right)$$
$$\log\left(1 - e^{-sN_0\log p + 2\pi i\left(\frac{k_1}{N_2'} + \frac{k_2}{N_1'}\right)}\right).$$

This equals to the sum of

$$(3.7) \quad -\left(1 - \frac{isN_0 \log p}{2\pi}\right) \sum_{n=1}^{\infty} \frac{p^{-sN_0 n}}{n}$$

$$\sum_{k_1=0}^{N_2'-1} e^{2\pi i \frac{nk_1}{N_2'}} \sum_{k_2=0}^{N_1'-1} e^{2\pi i \frac{nk_2}{N_1'}}$$

and

(3.8)

$$\sum_{n=1}^{\infty} \frac{p^{-sN_0 n}}{n} \sum_{k_1=0}^{N_2'-1} \sum_{k_2=0}^{N_1'-1} \left(\frac{k_1}{N_2'} + \frac{k_2}{N_1'}\right) e^{2\pi i \left(\frac{k_1}{N_2'} + \frac{k_2}{N_1'}\right) n}.$$

The first part (3.7) agrees to

$$\left(\frac{isN_0 \log p}{2\pi} - 1\right) \sum_{n=1}^{\infty} \frac{p^{-sN_0 n N_1' N_2'}}{n}.$$

Calculation of the remaining part (3.8) is carried out as follows:

$$\sum_{n=1}^{\infty} \frac{p^{-sN_0 n}}{n} \left(\sum_{k_1=0}^{N_2'-1} \frac{k_1}{N_2'} e^{2\pi i \frac{k_1 n}{N_2'}} \sum_{k_2=0}^{N_1'-1} e^{2\pi i \frac{k_2 n}{N_1'}}\right.$$

$$\left. + \sum_{k_2=0}^{N_1'-1} \frac{k_2}{N_1'} e^{2\pi i \frac{k_2 n}{N_1'}} \sum_{k_1=0}^{N_2'-1} e^{2\pi i \frac{k_1 n}{N_2'}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{p^{-sn N_0 N_1'}}{n N_1'} \sum_{k_1=0}^{N_2'-1} \frac{k_1}{N_2'} e^{2\pi i \frac{k_1 n N_1'}{N_2'}} N_1'$$

$$+ \sum_{n=1}^{\infty} \frac{p^{-sn N_0 N_2'}}{n N_2'} \sum_{k_2=0}^{N_1'-1} \frac{k_2}{N_1'} e^{2\pi i \frac{k_2 n N_2'}{N_1'}} N_2'$$

$$= \sum_{n=1}^{\infty} \frac{p^{-sn N_0 N_1'}}{n N_2'} \sum_{k_1=0}^{N_2'-1} k_1 e^{2\pi i \frac{k_1 n N_1'}{N_2'}}$$

$$+ \sum_{n=1}^{\infty} \frac{p^{-sn N_0 N_2'}}{n N_1'} \sum_{k_2=0}^{N_1'-1} k_2 e^{2\pi i \frac{k_2 n N_2'}{N_1'}}$$

$$= \sum_{n=1}^{\infty} \frac{p^{-sn N_1}}{n} f_1(n) + \sum_{n=1}^{\infty} \frac{p^{-sn N_2}}{n} f_2(n)$$

by Lemma 3.3.   □

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